Aerodynamic Characteristics of the Aircraft Wing

Anna Nazarenko

Department of Atomic Physics

Sofia University “St. Kliment Ohridski”

Supervisor

Assoc. Prof. Dr. Tsvetan Vetsov

Thesis for obtaining the educational qualification degree of **Bachelor**

Bachelor of Science in Nuclear and Particle Physics

September, 2025

Acknowledgements

I would like to express my profound gratitude to my supervisor for his exemplary guidance, insightful advice, and unwavering support throughout the development of this thesis. I am also grateful to my tutors over the years for their dedicated instruction and constructive feedback. Appreciation is further conveyed to colleagues for their valuable discussions and assistance, which greatly facilitated the successful completion of this work.

Abstract

This thesis focuses on the aerodynamic flow around airfoils using the Joukowski transformation, a complex-variable conformal mapping technique that transforms a circle into a symmetric or cambered airfoil shape. By analyzing potential flow around a circle and applying the Joukowski map, the behavior of ideal, inviscid flow around an airfoil can be studied analytically. This approach provides insight into fundamental aerodynamic quantities such as lift and pressure distribution and serves as a classical foundation for more advanced computational and experimental methods.

Contents

Chapter 1

Introduction

1.1 Overview of fluid mechanics

Fluid mechanics is a core area of classical physics that examines the behavior of fluids – liquids and gases – and the forces they exert on immersed objects. It offers the theoretical foundation for understanding fluid motion and interactions with solid surfaces, underpinning applications in aerodynamics, meteorology, oceanography, astrophysics, and numerous other disciplines.

The subject is broadly divided into fluid statics, which deals with fluids at rest, and fluid dynamics, which focuses on fluids in motion. Central to fluid mechanics is the continuum assumption, treating fluids as continuous media, which enables the use of differential equations to model their behavior. This leads to set of practically important equations such as the Navier[1]-Stokes[2] equations, which describe the motion of viscous[3], Newtonian fluids[4] under conservation of mass and momentum.

Key concepts include pressure, viscosity, flow regimes (laminar vs. turbulent), and the Reynolds number, which characterizes the relative importance of inertial and viscous effects. These principles

[1] Claude-Louis Navier (1785–1836) was a French engineer and physicist who contributed to elasticity theory and fluid mechanics. He introduced molecular considerations into continuum mechanics and formulated the first version of the equations governing viscous fluid flow, which now bear his name.

[2] George Gabriel Stokes (1819–1903) was an Irish mathematician and physicist who extended Navier's work on fluid motion. He established the modern form of the Navier–Stokes equations and made fundamental contributions to hydrodynamics, optics, and mathematical physics.

[3] A viscous fluid is a fluid that resists motion due to internal friction between its layers. This resistance is quantified by a property called viscosity, which measures how easily a fluid flows.

[4] Newtonian fluids are fluids whose viscosity remains constant regardless of the applied shear stress or shear rate. In other words, the relationship between shear stress and the rate of strain (deformation) is linear, and the fluid flows consistently under force. Water is an example of such a fluid.

support both theoretical and computational approaches in analyzing complex flow phenomena.

1.2 Aerodynamics as part of fluid mechanics

A major subfield of fluid mechanics is aerodynamics[1], which studies the gases in motion and their interaction with solid bodies, particularly in the context of lift, drag, and flow separation. Aerodynamics plays a central role in the design and analysis of aircraft wings (airfoils), wind turbines, and high-speed vehicles.

To determine the aerodynamic forces acting on an airplane or its components, it is necessary to solve the equations that govern the airflow around the vehicle. These airflow solutions can be formulated from the perspective of a ground-based observer or from the viewpoint of the pilot. As dictated by the laws of physics, both observers must arrive at equivalent results.

To a ground-based observer, the airplane is flying through a mass of air that is essentially at rest (assuming no wind). As the airplane moves, it accelerates and decelerates the surrounding air particles. The reaction of these particles to the imposed acceleration generates a force on the airplane. In this case, the flow field description in the ground-based frame of reference is time-dependent, representing an unsteady flow.

In the reference frame of the pilot, the airflow moves relative to the aircraft and adjusts according to the vehicle's geometry. If the aircraft maintains constant altitude and velocity, the velocity and thermodynamic properties of the airflow at any fixed point relative to the aircraft remain invariant with time. Consequently, the governing equations are typically more tractable when formulated in the vehicle-fixed (or pilot-fixed) reference frame, rather than in the ground-based coordinate system. Thus, many aerodynamic problems are formulated by modeling the flow of a fluid stream past a stationary body, where the frames of reference corresponding to the ground-based observer and the vehicle-fixed observer are related through a Galilean transformation[2].

Due to the often complex nature of flow patterns, a comprehensive description of the resulting flow frequently requires a combination of experimental investigations, theoretical analyses, and computational simulations.

[1] Its liquid fluid counterpart is called hydrodynamics.

[2] Galilean transformations are the change of coordinates between two inertial reference frames moving at constant relative velocity v in classical (non-relativistic) mechanics. For motion along x-axis: $x' = x - vt$, $y' = y$, $z' = z$.

1.3 Structure of the thesis

This thesis focuses on the aerodynamic flow around airfoils using the Joukowski[1] transformation, a complex-variable conformal mapping technique that transforms a circle into a symmetric or cambered airfoil shape. By analyzing potential flow around a circle and applying the Joukowski map, the behavior of ideal, inviscid flow around an airfoil can be studied analytically. This approach provides insight into fundamental aerodynamic quantities such as lift and pressure distribution and serves as a classical foundation for more advanced computational and experimental methods.

The structure of the thesis is organized as follows:

Chapter 2: We introduce the fundamental equations of fluid mechanics, derived from three basic physical principles: conservation of mass (continuity equation), conservation of energy (the first law of thermodynamics), and conservation of linear momentum (Newton's second law). This system is further complemented by the equation of state and the second law of thermodynamics for non-equilibrium conditions. We also derive both the integral and local differential forms of the governing equations under the simplifying assumptions of steady, incompressible flow with constant density ρ and no external body forces ($\vec{f} = 0$).

Chapter 3: We examine the two-dimensional, ideal, steady, uniform flow past a circular cylinder of radius R in the classical Kutta–Joukowski framework. Key flow quantities are derived, including the velocity potential, stream function, pressure distribution, and the resulting lift and drag forces.

Chapter 4: We extend the analysis to the two-dimensional Joukowski airfoil theory using complex variables. In this framework, we describe potential (irrotational and incompressible) flow around a circle displaced from the origin in the complex plane. By applying the Joukowski transformation – a conformal mapping that converts a circle into a symmetric or cambered airfoil shape – we analytically investigate the behavior of ideal, inviscid flow around an airfoil.

Chapter 5: We provide an effective description and analysis of a general airfoil-shaped wing. To illustrate the practical relevance of the theory, we present simulated velocity and pressure distributions around a real Boeing 737-800 wing, which support and validate the theoretical results obtained

[1] Nikolai Yegorovich Zhukovsky (1847–1921), often transliterated as Joukowski, was a Russian scientist regarded as the founder of Russian aerodynamics. He independently derived the lift theorem and pioneered the use of conformal mapping to transform circles into airfoil shapes, laying the foundation for modern airfoil theory. His institute in Moscow later became central to the development of Soviet aeronautics.

in the earlier chapters regarding lift generation.

Chapter 6: We conclude the thesis by summarizing the main findings and their implications and present some future prospects.

Chapter 2

Fundamentals of Fluid Mechanics

Chapter Objectives: This chapter focuses on deriving the key equations of fluid dynamics and developing a clear understanding of the underlying physical laws. We will use the following standard references on the subject [1–8].

2.1 Basic equations of fluid mechanics

The fluid motion is generally determined by three fundamental physical laws:

1. **Conservation of mass**, expressed by the continuity equation.

2. **Conservation of linear momentum** (Navier-Stokes equations), based on Newton's[1] second law of motion.

3. **Conservation of energy**, as stated in the first law of thermodynamics.

The basic equations stemming from these laws of fluid mechanics describe how fluids move and how forces act on them. The most commonly used equations are:

1. **Continuity equation** (conservation of mass). If ρ is the fluid mass density and $\vec{u}$ is the

[1] Isaac Newton (1642–1727) was an English mathematician, physicist, and natural philosopher whose Principia Mathematica laid the foundations of classical mechanics. In fluid mechanics, he introduced the concept of a linear relationship between shear stress and velocity gradient, defining what is now called a Newtonian fluid.

velocity vector[1], then the continuity equation ensures that mass is conserved within a fluid flow. There are two general cases:

- *For incompressible flow*[2] ($\partial_t \rho = 0$):

$$\nabla \cdot \vec{u} = 0, \quad \nabla_i u^i = 0. \tag{2.1}$$

- *For compressible flow*[3]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad \partial_t \rho + \nabla_i (\rho u^i) = 0, \tag{2.2}$$

where ∇_i is the covariant derivative[4] (accounts for curvature) with respect to a metric tensor[5] g^{ij} and an affine connection[6] (Christoffel symbols) Γ^j_{ik}, i.e.

$$\nabla_i u^j = \partial_i u^j + \Gamma^j_{ik} u^k, \quad \Gamma^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{li} - \partial_l g_{ik}). \tag{2.3}$$

2. **Navier-Stokes equations** (conservation of momentum):

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \nabla \cdot \hat{\tau} + \rho \vec{f}, \tag{2.4}$$

$$\rho \left(\partial_t u^i + u^j \nabla_j u^i \right) = -\nabla^i p + \nabla_j \tau^{ij} + \rho f^i, \tag{2.5}$$

[1]For the components of the local velocity we will use the following notations: $\vec{u} = (u, v, w)^T$. Obviously the velocity and its components dependent on the point in space and time $\vec{u} = \vec{u}(\vec{x}, t)$, i.e. it is a local vector field. Another example is the mass density $\rho = \rho(\vec{x}, t)$, which is a scalar vector field.

[2]Incompressible flow refers to the assumption that a fluid's density remains constant during motion, so that the continuity equation reduces to $\nabla \cdot \vec{u} = 0$. This is exact for liquids and a good approximation for gases at low Mach numbers ($M \lesssim 0.3$), where density variations are negligible.

[3]Compressible flow refers to fluid motion in which density variations are significant and cannot be neglected. It is common in gases at high speeds or under strong pressure/temperature changes, where the full continuity and energy equations must be used.

[4]Covariant derivative is a generalization of the usual derivative to curved spaces, allowing the differentiation of vectors and tensors in a way that respects the geometry. It accounts for changes in both the components and the basis, and is fundamental in differential geometry and general relativity.

[5]Metric tensor is a fundamental object in differential geometry that defines distances and angles on a manifold. It allows one to compute lengths of curves, angles between vectors, and volumes, and underlies the formulation of general relativity.

[6]Affine connection is a geometric object that defines how vectors are transported and differentiated along curves on a manifold. It specifies the rules for parallel transport and determines the covariant derivative, playing a central role in differential geometry and general relativity.

where p is pressure, $\vec{f}$ is body force per unit mass (e.g., gravity, electromagnetic force etc.), and $\hat{\tau}$ is the viscous stress tensor[1]:

$$\tau^{ij} = \mu\Big(\nabla^i u^j + \nabla^j u^i - \frac{2}{3} g^{ij}\nabla_k u^k\Big) + \zeta g^{ij}\nabla_k u^k, \tag{2.6}$$

with the dynamic viscosity μ and the bulk viscosity[2] ζ. Equations (2.4) and (2.5) are a fluid version of Newton's second law of motion[3].

3. **Conservation of energy** (the first law of thermodynamics): for compressible flows one has:

$$\rho\left(\frac{\partial \varepsilon}{\partial t} + \vec{u}\cdot\nabla\varepsilon\right) = -p(\nabla\cdot\vec{u}) + \Phi + \nabla\cdot(\kappa\nabla T) + \rho\dot{q}, \tag{2.7}$$

$$\rho\Big(\partial_t\varepsilon + u^j\nabla_j\varepsilon\Big) = -p\nabla_j u^j + \Phi + \nabla_i(\kappa\nabla^i T) + \rho\dot{q}, \tag{2.8}$$

where ε is the internal energy per unit mass, T is temperature, κ is thermal conductivity[4], $\Phi = \hat{\tau} : \nabla\vec{u} = \tau^{ij}\nabla_i u_j$ represents viscous dissipation function[5], and $\dot{q}$ is the heat-generation

[1] Viscous stress tensor $\hat{\tau}$ is a second-order tensor that describes the internal frictional forces in a fluid arising from gradients in the velocity field. Its components τ^{ij} quantify how shear and normal stresses develop in response to the rate of deformation, allowing viscous effects to be included in the momentum balance, as in the Navier–Stokes equations. For a Newtonian fluid, it is proportional to the strain rate tensor $\hat{S}$, with the proportionality given by the fluid's viscosity μ as in Eq. (2.35).

[2] Dynamic viscosity μ and bulk viscosity ζ are both measures of a fluid's resistance to flow, but they describe resistance to different types of deformation. Dynamic viscosity (resistance to shear deformation) μ, often simply called viscosity or shear viscosity, quantifies a fluid's resistance to shearing forces. Shear occurs when adjacent layers of a fluid move relative to each other, like when you spread honey with a knife, or when fluid flows through a pipe, with layers near the center moving faster than those near the walls. Bulk viscosity ζ (resistance to volume change, e.g. compression or expansion), also known as volume viscosity or second viscosity, quantifies a fluid's resistance to uniform compression or expansion (dilation). It describes the irreversible loss of mechanical energy when the fluid's volume changes.

[3] Newton's second law states that the net force acting on a body is equal to the rate of change of its linear momentum, $\vec{F} = \frac{d\vec{p}}{dt}(= m\vec{a}$ for constant mass). It provides the fundamental relationship between forces and motion, forming the basis of classical mechanics and governing both particle dynamics and continuum systems, including fluids and solids.

[4] Thermal conductivity is a physical property of a material that measures its ability to transfer heat through conduction. It defines the proportionality between the heat flux $\vec{q}$ and the temperature gradient ∇T via Fourier's law, $\vec{q} = -\kappa\nabla T$, where κ is the thermal conductivity. High values of κ correspond to materials that efficiently conduct heat, such as metals, while low values correspond to insulators.

[5] Viscous dissipation function Φ is a scalar quantity that measures the rate at which the mechanical energy of a fluid is irreversibly converted into thermal energy due to viscous stresses. It depends on the velocity gradient tensor $\nabla\vec{u}$ and the viscous stress tensor $\hat{\tau}$, capturing the work done by internal friction during shear and volumetric deformations. The viscious dissipation function Φ appears explicitly in the energy equation (2.7) of viscous flows, influencing temperature rise in high-viscosity fluids, boundary layers, and regions of strong shear, and plays a crucial role in thermofluid analyses where viscous heating is non-negligible.

rate per unit mass (W/kg) accounting for external or internal heat sources (e.g., radiation, chemical reactions etc.) This equation represents the work done by pressure, heat conduction, and viscous effects.

4. **Equation of state** (thermodynamic relation): An equation of state (EoS) is a thermodynamic equation that relates several state variables (or thermodynamic variables) that describe the state of matter under a given set of physical conditions. Essentially, it's a mathematical model that links fundamental properties of a substance, such as: pressure p, volume V, temperature T, number of moles n etc. The general form of a thermal equation of state is often written as:

$$f(p, V, T, n, ...) = 0, \tag{2.9}$$

e.g. for an ideal gas one has $p = \rho RT$, where R is the universal gas constant.

5. **Entropy inequality** (the second law of thermodynamics):

$$\rho \left(\frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s \right) \geq \frac{1}{T} \left(\Phi + \frac{k(\nabla T)^2}{T} \right), \tag{2.10}$$

$$\rho \left(\partial_t s + u^j \nabla_j s \right) \geq \frac{1}{T} \left(\Phi + \frac{k(\nabla_i T)^2}{T} \right), \tag{2.11}$$

where s is the specific entropy per unit mass. It expresses that entropy production is non-negative. The inequality describes real (irreversible) processes, while equality can be realized in ideal flows (perfect fluid, reversible adiabatic motion etc.).

The set of equations above form the foundation for both theoretical analysis and computational fluid dynamics (CFD). Depending on the problem (e.g., incompressible vs. compressible, laminar[1] vs. turbulent[2]), simplifications may be applied.

[1]Laminar flow is a flow regime in which fluid moves in smooth, orderly layers with minimal mixing between them. Velocity at each point remains steady in time, and the motion is dominated by viscous forces rather than inertia, typically occurring at low Reynolds numbers.

[2]Turbulent flow is a chaotic flow regime characterized by irregular fluctuations in velocity and pressure, strong mixing, and the dominance of inertial forces over viscous forces. It typically occurs at high Reynolds numbers.

2.2 The importance of the Navier-Stokes equations

The set of fluid mechanics equations (2.1)-(2.7) are partial differential equations that describe the motion of viscous fluids. Named after Claude-Louis Navier and George Gabriel Stokes, they were developed between 1822 and 1850 through gradual theoretical advances.

These equations express momentum conservation in Newtonian fluids and incorporate mass conservation. They derive from Newton's second law, assuming fluid stress combines a pressure term and a viscous term proportional to velocity gradients – capturing the behavior of viscous flow. Unlike the Euler[1] equations (A.2), which model inviscid flow, the Navier-Stokes equations include viscosity, making them elliptic[2] and more analytically tractable, though less mathematically structured.

Widely applied in science and engineering, the Navier-Stokes equations model weather patterns, ocean currents, pipe flow, and aerodynamics. They support the design of vehicles, power systems, and medical devices, and are crucial in fields like pollution analysis and magnetohydrodynamics when paired with Maxwell's equations.

Mathematically, they pose a major unsolved problem: whether smooth, bounded solutions exist in three dimensions. This "Navier-Stokes existence and smoothness" problem is one of the Clay Mathematics Institute's seven Millennium Prize Problems, with a $1 million reward for a solution or counterexample[3].

In what follows we will derive equations (2.2), (2.4) and (2.7), where, for clarity, certain simplifying assumptions will also be used.

[1]Leonhard Euler (1707–1783) was a Swiss mathematician and physicist who made foundational contributions to fluid mechanics, among many other fields. He formulated the Euler equations (A.2), which describe the motion of an ideal, inviscid fluid, laying the groundwork for modern theoretical hydrodynamics.

[2]Elliptic PDEs are a class of partial differential equations that model steady-state or equilibrium situations where the solution is influenced by conditions throughout the domain. They are characterized by the absence of real characteristic lines and typically yield smooth solutions. In fluid mechanics, elliptic PDEs arise in incompressible, irrotational flows, where the velocity potential satisfies Laplace's equation. Their solutions depend strongly on boundary conditions, reflecting the global influence of the domain.

[3]Navier–Stokes Millennium Problem: The Clay Mathematics Institute has listed the Navier–Stokes equations as one of its seven Millennium Prize Problems. The challenge is to prove or disprove whether smooth, globally defined solutions always exist in three dimensions for incompressible flows, or whether singularities (blow-ups) can develop in finite time. Solving this problem carries a one-million-dollar prize and remains a central open question in mathematical fluid mechanics.

2.3 Continuity equation and conservation of mass

Let us consider a small, two-dimensional rectangle in the xy-plane through which a fluid flows. The rectangle's faces are imaginary and do not impede the flow. Let u and v denote the components of the fluid velocity in the x- and y-directions, respectively. According to the principle of mass conservation, the net mass outflow through the surface enclosing the rectangle must equal the rate of decrease of mass within the volume.

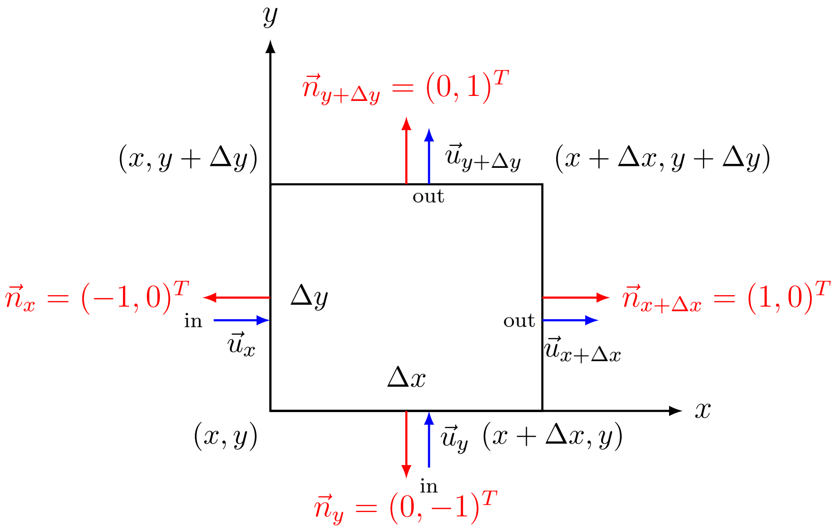


Figure 2.1 Control volume (CV) with outward normals (red) and velocity field (blue). Velocity vector indicate the mass inflow/outflow. For simplicity, the velocity vectors in the figure are drawn normal to each face, but in general, their direction is arbitrary; thus, at each face the velocity retains both components, $\vec{u} = (u, v)^T$. However, when computing the mass flux through a given face—e.g., the left face—only the velocity component normal to that face (here, u) contributes, since the tangential component (v) does not affect the flux.

To determine the mass flow rate $\dot{m} = \frac{dm}{dt}$ through the surface of the rectangle, we analyze the fluid flow across each face of the rectangle. The rectangular CV in Fig. 2.1 consists of four faces:

- Inlet (left face) at x, there ia a normal outward vector $\vec{n}_x = (-1,0)^T$, and an inward velocity $\vec{u}_x = (u_x, v_x)^T = \big(u(x,y), v(x, y_{\text{face}})\big)^T$. Note that the vertical component $v_x = v(x, y_{\text{face}})$ is tangential to the face and does not contribute, thus only the horizontal flux $u_x = u(x,y)$ matters.

- Outlet (right face) at $x+\Delta x$, there is a normal outward vector $\vec{n}_{x+\Delta x} = (1,0)^T$, and an outward velocity $\vec{u}_{x+\Delta x} = (u_{x+\Delta x}, v_{x+\Delta x})^T = \big(u(x+\Delta x, y), v(x+\Delta x, y_{\text{face}})\big)^T$.

- Bottom face at y, there is a normal outward vector $\vec{n}_y = (0,-1)^T$, and an inward velocity $\vec{u}_y = (u_y, v_y)^T = \big(u(x_{\text{face}}, y), v(x,y)\big)^T$.
- Top face at $y+\Delta y$, there is a normal outward vector $\vec{n}_{y+\Delta y} = (0,1)^T$, and an outward velocity $\vec{u}_{y+\Delta y} = (u_{y+\Delta y}, v_{y+\Delta y})^T = \big(u(x_{\text{face}}, y+\Delta y), v(x, y+\Delta y)\big)^T$.

The mass flow rate $\dot{m}$ is given by the product of density ρ, velocity (u or v), and the cross-sectional area (Δx or Δy) through which the fluid flows:

$$\dot{m} = \rho \times \text{velocity} \times \text{cross-sectional area}. \tag{2.12}$$

Let us first write this quantity in the x-direction (left and right faces). Through the left face we have an inflow at x:

$$\dot{m}_{x,\text{in}} = \rho(x,y)\, u(x,y)\, \Delta y = \rho u \Delta y. \tag{2.13}$$

The outflow through the right face at $(x+\Delta x)$ is:

$$\dot{m}_{x,\text{out}} = \rho(x+\Delta x, y)\, u(x+\Delta x, y)\, \Delta y = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x + \mathcal{O}(\Delta x^2)\right)\Delta y, \tag{2.14}$$

where we used a Taylor expansion for small Δx up to first power. The net flow in x-direction is then given by:

$$\Delta\dot{m}_x = \dot{m}_{x,\text{in}} - \dot{m}_{x,\text{out}} = \rho u \Delta y - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x + \mathcal{O}(\Delta x^2)\right)\Delta y \approx -\frac{\partial(\rho u)}{\partial x}\Delta x \Delta y. \tag{2.15}$$

The flow in the y-direction passes through the bottom and the top faces. The inflow through the bottom face at y is:

$$\dot{m}_{y,\text{in}} = \rho(x,y)\, v(x,y)\, \Delta x = \rho v \Delta x. \tag{2.16}$$

The outflow through the top face at $y+\Delta y$ is given by:

$$\dot{m}_{y,\text{out}} = \rho(x, y+\Delta y)\, v(x, y+\Delta y)\, \Delta x = \left(\rho v + \frac{\partial(\rho v)}{\partial y}\Delta y + \mathcal{O}(\Delta y^2)\right)\Delta x, \tag{2.17}$$

where we used a Taylor expansion for small Δy up to first power. The net flow in y-direction is thus:

$$\Delta \dot{m}_y = \dot{m}_{y,\text{in}} - \dot{m}_{y,\text{out}} = \rho v \Delta x - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\Delta y + \mathcal{O}(\Delta y^2)\right)\Delta x \approx -\frac{\partial(\rho v)}{\partial y}\Delta x \Delta y. \tag{2.18}$$

Finally, the total mass flow rate from both directions yields:

$$\dot{m}_{\text{total}} = \Delta \dot{m}_x + \Delta \dot{m}_y = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right)\Delta x \Delta y = -\text{div}(\rho \vec{u})\Delta x \Delta y, \tag{2.19}$$

where $\vec{u} = (u, v)$ is the velocity vector field of the fluid. The formula above represents the divergence of the mass flux $\rho\vec{u}$. If $\text{div}(\rho\vec{u}) > 0$, there is a net outflow (mass leaving the rectangle). If $\text{div}(\rho\vec{u}) < 0$, there is a net inflow (mass entering the rectangle). For the mass to be conserved (no accumulation in the rectangle), the net mass flow rate must be zero:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{or} \quad \partial_t \rho + \text{div}(\rho \vec{v}) = 0, \tag{2.20}$$

where we also included the flow in the z-direction. This is the mass continuity equation, where $\partial_t \rho$ accounts for any time-dependent changes in density and $\vec{u} = (u, v, w)$ is the velocity vector in three dimensions. In steady flow, $\partial_t \rho = 0$, and Eq. (2.20) simplifies to:

$$\text{div}(\rho \vec{u}) = 0. \tag{2.21}$$

Insisting further on incompressible flow $\rho = const$, one has $\nabla \cdot \vec{u} = 0$.

The next step towards deriving the full set of equations in fluid mechanics is the Reynolds Transport Theorem (RTT).

2.4 Reynolds Transport Theorem (RTT)

The Reynolds Transport Theorem provides a crucial link between the analysis of a system of fixed mass (a Lagrangian approach) and a fixed region in space, known as a control volume (an Eulerian approach). It is a fundamental principle in fluid mechanics used to derive the conservation laws for mass, momentum, and energy for a control volume.

2.4.1 Derivation of RTT

We will derive the RTT for a fixed control volume:

$$\dot{\Phi} = \frac{d\Phi_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial \phi}{\partial t}\, dV + \int_{\text{CS}} (\vec{u} \cdot \vec{n}) \phi \, dA. \tag{2.22}$$

This theorem states that the rate of change $\dot{\Phi}$ of an extensive property Φ for a system is equal to the rate of change $\partial_t \phi$ of its density ϕ within the control volume plus the net flux of that property out of the control surface. Before we prove this theorem, let's define our key terms:

- **System:** A specific quantity of mass that is followed as it moves and deforms. The mass within a system is constant.
- **Control volume (CV):** A fixed region in space through which the fluid flows. Its shape and size do not change with time.
- **Control surface (CS):** The boundary surface of the control volume.
- **Intensive property** $\phi(\vec{x}, t)$: Property of the fluid per unit volume, mass or charge.
- **Extensive property (global charge)** $\Phi(t)$: The total amount of the scalar property within a given volume, calculated by integrating the intensive property ϕ over that volume.

Our goal is to relate the rate of change $\dot{\Phi}$ of an extensive property Φ for a system to the changes occurring within a fixed control volume. For this purpose, let's consider a system that, at time t, exactly coincides with our fixed control volume. The total amount of the property Φ_{sys} in the system at time t is the same as that in the control volume at that instant:

$$\Phi_{\text{sys}}(t) = \Phi_{\text{CV}}(t) = \int_{\text{CV}} \phi(\vec{x}, t)\, dV. \tag{2.23}$$

At a later time, $t + \Delta t$, the system has moved. It now occupies a new region, which can be described as the original control volume minus the part the system has left (region I) and plus the new region it has entered (region II). The amount of the property Φ in the system at time $t + \Delta t$ is:

$$\Phi_{\text{sys}}(t + \Delta t) = \Phi_{\text{CV}}(t + \Delta t) - \Phi_{\text{I}}(t + \Delta t) + \Phi_{\text{II}}(t + \Delta t). \tag{2.24}$$

The rate of change $\dot{\Phi}_{\text{sys}}$ of the extensive property Φ_{sys} for the system is given by the definition of the derivative:

$$\frac{d\Phi_{\text{sys}}}{dt} = \lim_{\Delta t \to 0} \frac{\Phi_{\text{sys}}(t + \Delta t) - \Phi_{\text{sys}}(t)}{\Delta t}. \tag{2.25}$$

Substituting our expression (2.24) from the previous step, we find:

$$\begin{aligned} \frac{d\Phi_{\text{sys}}}{dt} &= \lim_{\Delta t \to 0} \frac{\big[\Phi_{\text{CV}}(t + \Delta t) - \Phi_{\text{I}}(t + \Delta t) + \Phi_{\text{II}}(t + \Delta t)\big] - \Phi_{\text{CV}}(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Phi_{\text{CV}}(t + \Delta t) - \Phi_{\text{CV}}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Phi_{\text{II}}(t + \Delta t) - \Phi_{\text{I}}(t + \Delta t)}{\Delta t}. \end{aligned} \tag{2.26}$$

The first term on the right-hand side is the definition of the time derivative of the extensive property Φ within the control volume. Since the control volume is fixed, the derivative can be moved inside the integral of its density representation:

$$\lim_{\Delta t \to 0} \frac{\Phi_{\text{CV}}(t + \Delta t) - \Phi_{\text{CV}}(t)}{\Delta t} = \frac{d\Phi_{\text{CV}}}{dt} = \frac{d}{dt} \int_{\text{CV}} \phi(\vec{x}, t)\, dV = \int_{\text{CV}} \frac{\partial \phi}{\partial t}\, dV. \tag{2.27}$$

The second limit term in (2.26) represents the net rate of flow, or flux, of the property Φ out of the control volume. In order to understand it let $\vec{u}$ be the velocity vector of the fluid and $\vec{n}$ be the outward-pointing normal vector to the control surface element dA. The amount of the property Φ crossing an area dA in time Δt is $\phi(\vec{u} \cdot \vec{n})\Delta t\, dA$. The net rate of outflow across the entire control surface is the integral over the control surface:

$$\lim_{\Delta t \to 0} \frac{\Phi_{\text{II}}(t + \Delta t) - \Phi_{\text{I}}(t + \Delta t)}{\Delta t} = \int_{\text{CS}} \phi(\vec{u} \cdot \vec{n})\, dA. \tag{2.28}$$

Finally, inserting the expressions from (2.27) and (2.28) back into (2.26), one arrives at the **Reynolds Transport Theorem** for a fixed control volume:

$$\frac{d\Phi_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial \phi}{\partial t}\, dV + \int_{\text{CS}} \phi(\vec{u} \cdot \vec{n})\, dA. \tag{2.29}$$

2.4.2 RTT for momentum

We consider a steady, incompressible, two-dimensional flow through a rectangular control volume (CV) of dimensions $\Delta x \Delta y$ (Fig. 2.1). The goal is to derive the linear momentum equation in the x- and y-directions. For this purpose, our starting point will be the **Reynolds Transport Theorem** (RTT) (2.29) applied to the momentum $\vec{P}$. In this case, it relates the rate of change $\dot{\vec{P}}$ of momentum $\vec{P}$ in a CV to the momentum flux across its control surfaces (CS). For a fixed CV, the change $\dot{\vec{P}}$ in linear momentum $\vec{P}$ equals the sum of all forces $\vec{F}_{\text{total}}$ acting on the fluid element:

$$\vec{I} \equiv \vec{F}_{\text{total}} = \frac{d\vec{P}}{dt} = \frac{\partial}{\partial t} \int_{CV} \rho \vec{u} \, dV + \oint_{CS} \rho \vec{u} (\vec{u} \cdot \vec{n}) \, dS. \tag{2.30}$$

Note that $\rho \vec{u}$ is a momentum per unit volume (momentum density), which integrated over dV gives momentum (i.e $\rho \vec{u}$ is the ϕ in RTT). For steady flow, all partial derivatives along the time ∂_t, are 0, thus:

$$\vec{I} = \oint_{CS} \rho \vec{u} (\vec{u}.\vec{n}) \, dS = \vec{F}_{\text{body}} + \vec{F}_{\text{surface}}, \tag{2.31}$$

where $\vec{F}_{\text{body}}$ is the external force acting on the body (e.g. gravity), and $\vec{F}_{\text{surface}}$ is the force on the surfaces (pressure plus viscous (shear) stresses):

$$\vec{F}_{\text{body}} = \int_{CV} \rho \vec{f} \, dV, \quad \vec{F}_{\text{surface}} = \oint_{CS} \vec{T} \, dS. \tag{2.32}$$

Here the vector $\vec{T}$ is the stress vector (traction), which represents the surface force per unit area exerted on the control surface by the surrounding fluid, and is related to the Cauchy stress tensor $\hat{\sigma}$ through:

$$\vec{T} = \hat{\sigma} \cdot \vec{n} = -p \, \vec{n} + \hat{\tau} \cdot \vec{n}. \tag{2.33}$$

where $\vec{n}$ is the outward-pointing unit normal vector to the surface element dS. The stress tensor $\hat{\sigma}$ contains both pressure contribution $(-p\hat{I})$, and viscous (shear) stresses $(\hat{\tau})$:

$$\hat{\sigma} = -p\hat{I} + \hat{\tau}. \tag{2.34}$$

The vector $\vec{T}$ points in the direction of the fluid, i.e. it is pushing or pulling on the surface element. If $\vec{T}$ has a tangential component, it will try to slide the surface element along the surface (shear). If it has a normal component, it will push or pull perpendicular to the surface (pressure).

Shear stresses $\hat{\tau}$ are the tangential components of stress – forces per unit area (N/m^2) that act parallel to a material's surface, rather than perpendicular to it. For a Newtonian fluids[1], the viscous stress tensor are defined by the Newton law of viscosity:

$$\hat{\tau} = 2\mu\hat{S} = \mu\big[\nabla\vec{u} + (\nabla\vec{u})^T\big], \quad \tau^{ij} = \mu\big(\nabla^i u^j + \nabla^j u^i\big), \tag{2.35}$$

where $\hat{S}$ represents the symmetric (strain) part of the velocity gradient tensor decomposition into symmetric and antisymmetric part:

$$\nabla\vec{u} = \frac{1}{2}\big[\nabla\vec{u} + (\nabla\vec{u})^T\big] + \frac{1}{2}\big[\nabla\vec{u} - (\nabla\vec{u})^T\big] = \hat{S} + \hat{\Omega}. \tag{2.36}$$

Since for Newtonian fluids the rotational part $\hat{\Omega}$ is zero, then for instance, in $2d$ we can write:

$$\hat{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} = \mu \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{pmatrix}. \tag{2.37}$$

A more general expression for $\hat{\tau}$ is presented in Eq. (2.6), where contributions also from dilatations (stretching of the control volume) and curvature are included.

[1] Newtonian fluid is a fluid whose viscosity (resistance to flow) remains constant regardless of the rate at which it is deformed (shear rate). A classic example of a Newtonian fluid is water. If you pour water from a cup slowly or quickly, it flows with the same viscosity. In lab tests, the shear stress vs. shear rate graph for water is a straight line through the origin, meaning its viscosity is constant at around $\mu \approx 10^{-3}\,\mathrm{Pa}\cdot\mathrm{s}$. Non-Newtonian fluids, in contrast, have viscosities that change depending on the shear rate (e.g., ketchup, oobleck, toothpaste).

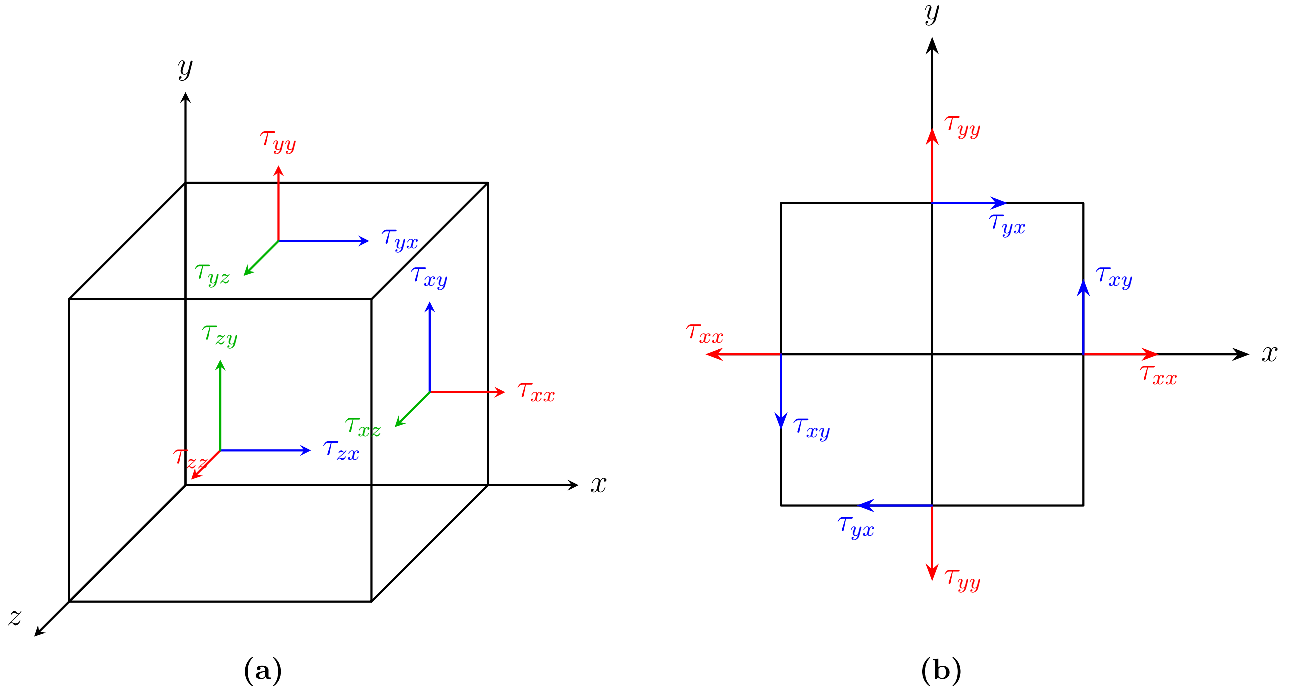


Figure 2.2 **(a)** Stress tensor ($3d$). **(b)** Stress tensor ($2d$).

2.5 Conservation of linear momentum

We will now apply RTT to the rectangular CV shown in Fig. 2.1. Assume the velocity components are $\vec{u} = (u, v)^T$, the flow is incompressible with constant density ρ, and there are no body forces ($\vec{f} = 0$) for simplicity. The momentum flux term (2.31) is evaluated on each face:

- Inlet (left face) at x with normal vector $\vec{n}_x = (-1, 0)^T$ and components of the velocity $\vec{u}_x = (u, v)^T_x$. The mass flux per unit area is given by: $\rho\vec{u}_x(\vec{u} \cdot \vec{n})_x = \rho(-u_x)\vec{u}_x = \rho(-u\,\vec{u})_x$. Therefore, the momentum flux (2.31) is ($\rho = const$):

$$\vec{I}_{\text{left}} \equiv \vec{I}_x = \int_y^{y+\Delta y} \rho(-u\,\vec{u})_x\,dy = -\rho\int_y^{y+\Delta y} \begin{pmatrix} u^2 \\ uv \end{pmatrix}_x dy \approx -\rho \begin{pmatrix} \overline{u}^2 \\ \overline{uv} \end{pmatrix}_x \Delta y. \tag{2.38}$$

Here, for small Δy we approximated the integral over the x face using the Euler forward (left endpoint) average[1]:

$$\int_y^{y+\Delta y} f(y)dy \approx \overline{f}(y)\Delta y. \tag{2.39}$$

• Outlet (right face, $x+\Delta x$) with normal vector $\vec{n}_{x+\Delta x} = (1,0)^T$ and components of the velocity $\vec{u}_{x+\Delta x} = (u_{x+\Delta x}, v_{x+\Delta x})^T$. The mass flux is: $\rho\vec{u}_{x+\Delta x}(\vec{u}\cdot\vec{n})_{x+\Delta x} = \rho\vec{u}_{x+\Delta x}u_{x+\Delta x} = \rho(u\,\vec{u})_{x+\Delta x}$, and the momentum flux is:

$$\vec{I}_{\text{right}} \equiv \vec{I}_{x+\Delta x} = \int_y^{y+\Delta y} \rho(u\,\vec{u})_{x+\Delta x}\,dy = \rho\int_y^{y+\Delta y} \begin{pmatrix} u^2 \\ uv \end{pmatrix}_{x+\Delta x} dy \approx \rho \begin{pmatrix} \overline{u}^2 \\ \overline{uv} \end{pmatrix}_{x+\Delta x} \Delta y. \tag{2.40}$$

• Bottom face (at y) with normal vector $\vec{n}_y = (0,-1)^T$ and components of the velocity $\vec{u}_y = (u_y, v_y)^T$. The mass flux is: $\rho\vec{u}_y(\vec{u}\cdot\vec{n})_y = \rho\vec{u}_y(-v_y) = \rho(-v\,\vec{u})_y$, and the momentum flux is:

$$\vec{I}_{\text{bottom}} \equiv \vec{I}_y = \int_x^{x+\Delta x} \rho(-v\,\vec{u})_y\,dx = -\rho\int_x^{x+\Delta x} \begin{pmatrix} uv \\ v^2 \end{pmatrix}_y dx \approx -\rho \begin{pmatrix} \overline{uv} \\ \overline{v}^2 \end{pmatrix}_y \Delta x. \tag{2.41}$$

• Top face ($y+\Delta y$) with normal vector $\vec{n}_{y+\Delta y} = (0,1)^T$ and components of the velocity $\vec{u}_{y+\Delta y} = (u_{y+\Delta y}, v_{y+\Delta y})^T$. The mass flux is: $\rho\vec{u}_{y+\Delta y}(\vec{u}\cdot\vec{n})_{y+\Delta y} = \rho\vec{u}_{y+\Delta y}\,v_{y+\Delta y} = \rho(v\,\vec{u})_{y+\Delta y}$, and the momentum flux is:

$$\vec{I}_{\text{top}} \equiv \vec{I}_{y+\Delta y} = \int_x^{x+\Delta x} \rho(v\,\vec{u})_{y+\Delta y}\,dx = \rho\int_x^{x+\Delta x} \begin{pmatrix} uv \\ v^2 \end{pmatrix}_{y+\Delta y} dx \approx \rho \begin{pmatrix} \overline{uv} \\ \overline{v}^2 \end{pmatrix}_{y+\Delta y} \Delta x. \tag{2.42}$$

The net momentum flux is obtained after combining the contributions of all faces:

$$\begin{aligned}\vec{I} = \oint_{CS} \rho\vec{u}(\vec{u}\cdot\vec{n})\,dS &= \vec{I}_{\text{left}} + \vec{I}_{\text{right}} + \vec{I}_{\text{bottom}} + \vec{I}_{\text{top}} \\ &= \begin{pmatrix} \rho\big(u^2_{x+\Delta x} - u^2_x\big)\Delta y + \rho\big[(uv)_{y+\Delta y} - (uv)_y\big]\Delta x \\ \rho\big[(uv)_{x+\Delta x} - (uv)_x\big]\Delta y + \rho\big(v^2_{y+\Delta y} - v^2_y\big)\Delta x \end{pmatrix}.\end{aligned} \tag{2.43}$$

Note that we dropped the overbar notation for the averages for simplicity.

[1]The approximation arises because we replace the integral of the generally varying function $f(y)$ over $[y, y+\Delta y]$ by the product of the interval length Δy and a representative value $\overline{f}(y)$. For example, using the left-endpoint (Euler forward) average, $\overline{f}(y) = f(y)$. This is exact only if $f(y)$ is constant; otherwise, it is an approximation, which improves as $\Delta y \to 0$.

2.5.1 Surface forces (pressure + viscous stresses)

The surface force $\vec{F}_{\text{surface}} = \oint_{CS} \vec{T}\, dS$ includes the pressure forces and viscous (shear) stresses, i.e.:

$$\vec{F}_{\text{surface}} = \oint_{CS} \vec{T}\, dS = \oint_{CS} \vec{T}_{\text{pressure}}\, dS + \oint_{CS} \vec{T}_{\text{stress}}\, dS = \vec{F}_{\text{pressure}} + \vec{F}_{\text{stress}}, \tag{2.44}$$

where

$$\vec{T}_{\text{pressure}} = -p\vec{n}, \quad \vec{T}_{\text{stress}} = \hat{\tau} \cdot \vec{n}. \tag{2.45}$$

- Let us calculate the **pressure force** first. Along the x-direction one has ($dS \approx \Delta y$):

$$-\oint p_x n_x\, dS - \oint p_{x+\Delta x} n_{x+\Delta x}\, dS \approx -\left(p_{x+\Delta x} - p_x\right) \Delta y, \tag{2.46}$$

where $n_x = -1$ and $n_{x+\Delta x} = 1$. Along y-direction ($dS \approx \Delta x$):

$$-\oint p_y n_y\, dS - \oint p_{y+\Delta y} n_{y+\Delta y}\, dS \approx -\left(p_{y+\Delta y} - p_y\right) \Delta x, \tag{2.47}$$

where $n_y = -1$ and $n_{y+\Delta y} = 1$. Therefore, the total pressure force is:

$$\vec{F}_{\text{pressure}} = \begin{pmatrix} -\left(p_{x+\Delta x} - p_x\right) \Delta y \\ -\left(p_{y+\Delta y} - p_y\right) \Delta x \end{pmatrix}. \tag{2.48}$$

Note that in the calculations above we also used the Euler average rule (2.39).

- To find the **viscous force** contribution we compute the traction on each face:

$$\vec{J}_{\text{left}} = \oint_{CS} (\hat{\tau} \cdot \vec{n})\, dS\big|_x = \int_y^{y+\Delta y} (\hat{\tau} \cdot \vec{n})\, dS\big|_x \approx (\hat{\tau} \cdot \vec{n})|_x \Delta y = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Delta y = \begin{pmatrix} -\tau_{xx}|_x \\ -\tau_{xy}|_x \end{pmatrix} \Delta y,$$

$$\vec{J}_{\text{right}} = \oint_{CS} (\hat{\tau} \cdot \vec{n})\, dS\big|_{x+\Delta x} = \int_y^{y+\Delta y} (\hat{\tau} \cdot \vec{n})\, dS\big|_{x+\Delta x} \approx (\hat{\tau} \cdot \vec{n})|_{x+\Delta x} \Delta y = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Delta y = \begin{pmatrix} \tau_{xx}|_{x+\Delta x} \\ \tau_{xy}|_{x+\Delta x} \end{pmatrix} \Delta y,$$

$$\vec{J}_{\text{bottom}} = \oint_{CS} (\hat{\tau} \cdot \vec{n})\, dS\big|_y = \int_x^{x+\Delta y} (\hat{\tau} \cdot \vec{n})\, dS\big|_y \approx (\hat{\tau} \cdot \vec{n})|_y \Delta x = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Delta x = \begin{pmatrix} -\tau_{xy}|_y \\ -\tau_{yy}|_y \end{pmatrix} \Delta x,$$

$$\vec{J}_{\text{top}} = \oint_{CS} (\hat{\tau} \cdot \vec{n})\, dS\big|_{y+\Delta y} = \int_x^{x+\Delta x} (\hat{\tau} \cdot \vec{n})\, dS\big|_{y+\Delta y} \approx (\hat{\tau} \cdot \vec{n})|_{y+\Delta y} \Delta x = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Delta x = \begin{pmatrix} \tau_{xy}|_{y+\Delta y} \\ \tau_{yy}|_{y+\Delta y} \end{pmatrix} \Delta x.$$

Note that in the calculations above we used the Euler average rule (2.39). The total viscous force is:

$$\begin{aligned} \vec{F}_{\text{stress}} &= \oint_{CS} \vec{T}_{\text{stress}}\, dS = \vec{J}_{\text{left}} + \vec{J}_{\text{right}} + \vec{J}_{\text{bottom}} + \vec{J}_{\text{top}} \\ &= \begin{pmatrix} (\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x)\Delta y + (\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y)\Delta x \\ (\tau_{xy}|_{x+\Delta x} - \tau_{xy}|_x)\Delta y + (\tau_{yy}|_{y+\Delta y} - \tau_{yy}|_y)\Delta x \end{pmatrix}. \end{aligned} \tag{2.49}$$

- We can now obtain the final integral momentum equations (2.31),

$$\vec{I} = \vec{F}_{\text{pressure}} + \vec{F}_{\text{stress}}, \tag{2.50}$$

by combining all terms from (2.43), (2.48) and (2.49). Therefore, the x-momentum equation is:

$$\begin{aligned} &\rho\big(u^2_{x+\Delta x} - u^2_x\big)\Delta y + \rho\big[(uv)_{y+\Delta y} - (uv)_y\big]\Delta x \\ &= -\left(p_{x+\Delta x} - p_x\right)\Delta y + (\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x)\Delta y + (\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y)\Delta x, \end{aligned} \tag{2.51}$$

and the y-momentum equation is:

$$\begin{aligned} &\rho\big[(uv)_{x+\Delta x} - (uv)_x\big]\Delta y + \rho\big(v^2_{y+\Delta y} - v^2_y\big)\Delta x \\ &= -\left(p_{y+\Delta y} - p_y\right)\Delta x + (\tau_{xy}|_{x+\Delta x} - \tau_{xy}|_x)\Delta y + (\tau_{yy}|_{y+\Delta y} - \tau_{yy}|_y)\Delta x. \end{aligned} \tag{2.52}$$

Let us divide these equations by $\Delta x \Delta y$:

$$\rho\frac{u^2_{x+\Delta x} - u^2_x}{\Delta x} + \rho\frac{(uv)_{y+\Delta y} - (uv)_y}{\Delta y} = -\frac{p_{x+\Delta x} - p_x}{\Delta x} + \frac{\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x}{\Delta x} + \frac{\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y}{\Delta y}, \tag{2.53}$$

$$\rho\frac{(uv)_{x+\Delta x} - (uv)_x}{\Delta x} + \rho\frac{v^2_{y+\Delta y} - v^2_y}{\Delta y} = -\frac{p_{y+\Delta y} - p_y}{\Delta y} + \frac{\tau_{xy}|_{x+\Delta x} - \tau_{xy}|_x}{\Delta x} + \frac{\tau_{yy}|_{y+\Delta y} - \tau_{yy}|_y}{\Delta y}. \tag{2.54}$$

Let us take the limits $\Delta x, \Delta y \to 0$ in the first Eq. (2.53) and use the definition of the derivative:

$$\rho\frac{u_{x+\Delta x}^2 - u_x^2}{\Delta x} \to \rho\frac{\partial}{\partial x}(u^2), \quad \rho\frac{(uv)_{y+\Delta y} - (uv)_y}{\Delta y} \to \rho\frac{\partial}{\partial y}(uv), \tag{2.55}$$

$$-\frac{p_{x+\Delta x} - p_x}{\Delta x} \to -\frac{\partial p}{\partial x}, \quad \frac{\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x}{\Delta x} \to \frac{\partial \tau_{xx}}{\partial x}, \quad \frac{\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y}{\Delta y} \to \frac{\partial \tau_{xy}}{\partial y}. \tag{2.56}$$

Hence the differential form of Eq. (2.53) is:

$$\rho\frac{\partial}{\partial x}(u^2) + \rho\frac{\partial}{\partial y}(uv) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}. \tag{2.57}$$

We can rewrite the left-hand side in more common form by the product-rule expansions:

$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}, \quad \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}, \tag{2.58}$$

hence:

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = 2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}. \tag{2.59}$$

Now invoke incompressibility ($\nabla \cdot \vec{u} = 0$), i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, which gives $u\frac{\partial v}{\partial y} = -u\frac{\partial u}{\partial x}$. Substituting:

$$2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} = 2u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}. \tag{2.60}$$

Thus Eq. (2.57) becomes:

$$\rho\Big(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}. \tag{2.61}$$

Let us now treat the viscous terms. For a Newtonian incompressible fluid with constant μ:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \quad \tau_{xy} = \mu\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big). \tag{2.62}$$

Next we differentiate:

$$\frac{\partial \tau_{xx}}{\partial x} = 2\mu\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial \tau_{xy}}{\partial y} = \mu\Big(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\Big). \tag{2.63}$$

Now we add up:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y}. \tag{2.64}$$

One can use incompressibility differentiated with respect to x:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = 0 \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}. \tag{2.65}$$

Substitute back into (2.64) to find:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} - \mu \frac{\partial^2 u}{\partial x^2} = \mu \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) = \mu \nabla^2 u. \tag{2.66}$$

Putting everything together yields the familiar x-momentum PDE:

$$\rho \Big(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \Big) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u. \tag{2.67}$$

Similarly, one can treat the y-momentum Eq. (2.54). Let us write them both here explicitly:

- The x-momentum equation:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{2.68}$$

- The y-momentum equation:

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \tag{2.69}$$

In conclusion, the integral momentum conservation for $2d$ flow through a rectangle involves momentum flux (convection) across all faces, pressure forces, and viscous stresses. In the limit of an infinitesimal CV, we recover the Navier-Stokes equations (2.68) and (2.69) for $2d$ incompressible flow.

2.5.2 Navier-Stokes equations for incompressible flow

We are ready to write the final form of the $3d$ Navier-Stokes equations for incompressible flow, which are the momentum conservation equations (2.68) and (2.69):

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right)=-\nabla p+\mu\nabla^2\vec{u}, \tag{2.70}$$

which we can write in components using $\vec{u}=(u,v,w)^T$:

$$x-\text{momentum}: \quad \rho\left(\frac{\partial u}{\partial t}+\vec{u}\cdot\nabla u\right)=-\frac{\partial p}{\partial x}+\mu\nabla^2 u, \tag{2.71}$$

$$y-\text{momentum}: \quad \rho\left(\frac{\partial v}{\partial t}+\vec{u}\cdot\nabla v\right)=-\frac{\partial p}{\partial y}+\mu\nabla^2 v, \tag{2.72}$$

$$z-\text{momentum}: \quad \rho\left(\frac{\partial w}{\partial t}+\vec{u}\cdot\nabla w\right)=-\frac{\partial p}{\partial z}+\mu\nabla^2 w. \tag{2.73}$$

Using the definition of the material derivative:

$$\frac{D\vec{u}}{Dt}=\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}, \tag{2.74}$$

we can write the NS equations above in a compact vector form:

$$\boxed{\rho\frac{D\vec{u}}{Dt}=-\nabla p+\mu\nabla^2\vec{u}.} \tag{2.75}$$

This formulation is coordinate-invariant and valid for any inertial reference frame. Note that for compressible flows, additional terms would appear, and ρ would vary spatially.

2.6 The first law of thermodynamics

The first law of thermodynamics states that the change in total energy E of a fluid system equals the heat Q added to the system minus the work W done by the system[1]:

$$dE = \delta Q - \delta W, \tag{2.76}$$

where δ is used to represent an inexact differential (process dependent quantity). We can divide by dt to get the time-dependent form of the first law:

$$\frac{dE}{dt} = \frac{\delta Q}{dt} - \frac{\delta W}{dt} = \dot{Q} - \dot{W}. \tag{2.77}$$

- The total energy of a fluid in a control volume is given by:

$$E(t) = \iiint_{\mathrm{CV}} \rho(\vec{x},t)\, \varepsilon(\vec{x},t)\, dV = \iiint_{\mathrm{CV}} \rho\varepsilon\, dV, \tag{2.78}$$

where ρ is the mass density, ε is energy per unit mass, and dV is an infinitesimal volume element. Since the CV is not fixed (it moves or deform with time), the time derivative cannot pass inside the integral and the rate of change of the total energy is:

$$\frac{dE}{dt} = \frac{d}{dt} \iiint_{\mathrm{CV}} (\rho\varepsilon)\, dV. \tag{2.79}$$

However, the Reynolds Transport Theorem (2.29) says that for a scalar quantity Φ and its density ϕ one has the relation:

$$\frac{d\Phi}{dt} = \int_{\mathrm{CV}} \frac{\partial \phi}{\partial t}\, dV + \int_{\mathrm{CS}} \phi(\vec{u}\cdot\vec{n})\, dA. \tag{2.80}$$

Applying for $\phi = \rho\varepsilon$ we can write:

$$\frac{dE}{dt} = \iiint_{\mathrm{CV}} \frac{\partial(\rho\varepsilon)}{\partial t}\, dV + \iint_{\mathrm{CS}} \rho\varepsilon(\vec{u}\cdot\vec{n})\, dA, \tag{2.81}$$

where CS is the control surface with infinitesimal surface element dA and an outward unit normal

[1] In this convention of the first law: heat is positive when entering the control volume, and work is positive when done by the system on the surroundings.

vector $\vec{n}$. The first term is the local (fixed point) rate of change of energy inside the volume. The second term accounts for the net rate at which energy is carried in or out by the fluid moving through the control surface.

- The heat transfer rate ($\dot{Q}$) into the CV is defined by[1]:

$$\dot{Q} = -\iint_{\mathrm{CS}} \vec{q} \cdot \vec{n}\, dA + \iiint_{\mathrm{CV}} \dot{q}_{\mathrm{v}}\, dV, \tag{2.82}$$

where $\vec{q} = -\kappa \nabla T$ (Fourier's law) is the heat flux vector, κ is the thermal conductivity, and $\dot{q}_{\mathrm{v}}$ is an internal heat generation also known as the volumetric heat generation rate (W/m^3).

- Finally, the work done by the fluid for unit time ($\dot{W}$) is[2]:

$$\dot{W} = \underbrace{\iint_{\mathrm{CS}} p(\vec{u} \cdot \vec{n})\, dA}_{\text{Pressure work}} - \underbrace{\iint_{\mathrm{CS}} \vec{u} \cdot (\hat{\tau} \cdot \vec{n})\, dA}_{\text{Viscous work}}. \tag{2.83}$$

Therefore, the integral form of the first law of thermodynamics for a control volume is:

$$\begin{aligned}\iiint_{\mathrm{CV}} \frac{\partial(\rho\varepsilon)}{\partial t}\, dV &+ \iint_{\mathrm{CS}} \rho\varepsilon(\vec{u} \cdot \vec{n})\, dA \\ &= \left(-\iint_{\mathrm{CS}} \vec{q} \cdot \vec{n}\, dA + \iiint_{\mathrm{CV}} \dot{q}_{\mathrm{v}}\, dV\right) - \left(\iint_{\mathrm{CS}} p(\vec{u} \cdot \vec{n})\, dA - \iint_{\mathrm{CS}} \vec{u} \cdot (\hat{\tau} \cdot \vec{n})\, dA\right). \end{aligned} \tag{2.84}$$

[1]The minus sign in front of the surface integral comes from the convention that $\vec{q} \cdot \vec{n}$ is the outward flux (positive when heat leaves the CV). Since $\vec{q} \cdot \vec{n}$ is the outward conductive flux, the surface term with a leading minus is the heat into the CV. The volumetric source $\dot{q}_v > 0$ adds heat inside – both are heat into the system

[2]Due to our convention of the first law $dE = \delta Q - \delta W$, the quantity $\dot{W}$ is the rate of work done by the system on its surroundings, thus the signs of the integrals are correct, otherwise if $\dot{W}$ is the work done on the system, the overall sign should be reversed. To see this start from the Cauchy stress $\hat{\sigma} = -p\hat{I} + \hat{\tau}$. The power of the surface traction acting **on** the control volume is:

$$\dot{W}_{\text{on CV}} = \iint_{\mathrm{CS}} \vec{u} \cdot (\hat{\sigma} \cdot \vec{n})\, dA = \iint_{\mathrm{CS}} \vec{u} \cdot (-p\vec{n} + \hat{\tau} \cdot \vec{n})\, dA = -\iint_{\mathrm{CS}} p(\vec{u} \cdot \vec{n})\, dA + \iint_{\mathrm{CS}} \vec{u} \cdot (\hat{\tau} \cdot \vec{n})\, dA.$$

So here the pressure contribution appears with a minus sign. However, in our convention of the first law $\dot{W}$ is the rate of work done **by** the system on the surroundings, then:

$$\dot{W} \;=\; -\dot{W}_{\text{on CV}} = \iint_{\mathrm{CS}} p(\vec{u} \cdot \vec{n})\, dA - \iint_{\mathrm{CS}} \vec{u} \cdot (\hat{\tau} \cdot \vec{n})\, dA.$$

We can now use the Divergence Theorem for an arbitrary vector field $\vec{F}(\vec{x},t)$,

$$\iint_{\text{CS}} (\vec{F}\cdot\vec{n})\, dA = \iiint_{\text{CV}} (\nabla\cdot\vec{F})\, dV, \tag{2.85}$$

to convert all surface integrals (CS) into volume integrals (CV), hence:

$$\iiint_{\text{CV}} \left[\frac{\partial(\rho\varepsilon)}{\partial t} + \nabla\cdot(\rho\varepsilon\vec{u})\right] dV = \iiint_{\text{CV}} \Big[-\nabla\cdot\vec{q} + \dot{q}_{\text{v}} - \nabla\cdot(p\vec{u}) + \nabla\cdot(\hat{\tau}\cdot\vec{u})\Big] dV. \tag{2.86}$$

Note that the conversion of the viscous work term is:

$$\iint_{\text{CS}} \vec{u}\cdot(\hat{\tau}\cdot\vec{n})\, dA = \iiint_{\text{CV}} \nabla\cdot(\hat{\tau}\cdot\vec{u})\, dV. \tag{2.87}$$

Since this integral equation must hold for any arbitrary control volume, the integrands must be equal. This gives the differential form of the energy equation (the first law of thermodynamics):

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \nabla\cdot(\rho\varepsilon\vec{u}) = -\nabla\cdot\vec{q} + \dot{q}_{\text{v}} - \nabla\cdot(p\vec{u}) + \nabla\cdot(\hat{\tau}\cdot\vec{u}). \tag{2.88}$$

For constant in space ρ and p one recovers Eq. (2.7):

$$\rho\left(\frac{\partial\varepsilon}{\partial t} + \vec{u}\cdot\nabla\varepsilon\right) = -p(\nabla\cdot\vec{u}) + \Phi + \nabla\cdot(\kappa\nabla T) + \rho\dot{q}, \tag{2.89}$$

where $\Phi = \nabla\cdot(\hat{\tau}\cdot\vec{u})$ represents the viscous dissipation function, $\vec{q} = -\kappa\nabla T$ (Fourier's law) is the heat flux vector, and $\dot{q}_{\text{v}} = \rho\dot{q}$ with $\dot{q}$being the internal heat-generation rate per unit mass (W/kg).

Note that the general expression is $\Phi = \hat{\tau} : \nabla\vec{u} = \tau^{ij}\nabla_i u_j$ differs from $\Phi = \nabla\cdot(\hat{\tau}\cdot\vec{u})$ in Eq. (2.89). However, using tensor calculus one has:

$$\nabla\cdot(\hat{\tau}\cdot\vec{u}) = \nabla_k(\tau^{kj}u_j) = (\nabla_k\tau^{kj})u_j + \tau^{kj}\nabla_k u_j, \tag{2.90}$$

hence:

$$\nabla\cdot(\hat{\tau}\cdot\vec{u}) = (\nabla\cdot\hat{\tau})\cdot\vec{u} + \hat{\tau} : (\nabla\vec{u})^T. \tag{2.91}$$

For Newtonian fluids, $\hat{\tau}$ is symmetric ($\tau^{ij} = \tau^{ji}$), which implies:

$$\hat{\tau} : (\nabla\vec{u})^T = \tau^{ij}(\nabla\vec{u})^T_{ji} = \tau^{ij}\nabla_j u_i. \tag{2.92}$$

Since $\hat{\tau}$ is symmetric, we can relabel indices $i \leftrightarrow j$:

$$\tau^{ij}\nabla_j u_i = \tau^{ji}\nabla_j u_i = \tau^{ij}\nabla_i u_j. \tag{2.93}$$

Additionally, $\nabla \cdot \hat{\tau} = 0$, when viscous dissipation is balanced by work done by viscous stresses at boundaries, with no net force on fluid elements (i.e. non-zero viscosity vs. zero net viscous force)[1]. Therefore, in this special case, $\Phi = \hat{\tau} : \nabla\vec{u} \sim \nabla \cdot (\hat{\tau} \cdot \vec{u})$.

In the following, we apply the fundamental equations of fluid mechanics and their solutions to analyze basic two-dimensional flows around a circular cylinder and a Joukowski airfoil.

[1] It signifies that viscous stresses are self-equilibrating. They cause local deformation (and dissipation) but no net force on fluid parcels.

Chapter 3

Axisymetric Airflow in 2d

Chapter objectives: We study a two-dimensional steady uniform flow past a circular cylinder of radius R in the so called Kutta–Joukowski setup. The aim is to derive key flow quantities, including the velocity potential, stream function, pressure distribution, and lift force. This chapter demonstrates the power and the applicability of fluid dynamics equations considered in the previously.

3.1 Basic assumptions and equations

3.1.1 Basic assumptions

We consider a $2d$, steady, incompressible, inviscid, irrotational flow around a circle (cylinder) of radius R with circulation Γ. The goal is to derive:

1. The velocity potential ϕ and stream function ψ.
2. The velocity field $\vec{u} = (u, v)$ on and around the cylinder
3. The pressure distribution p using Bernoulli's equation.
4. The lift force (Magnus force) using the Kutta-Joukowski theorem.

To obtain analytical solutions to the governing equations of fluid dynamics, we impose the following assumptions (by neglecting some effects for potential flow):

- No viscosity ($\mu = 0$): Removes the Navier-Stokes viscous term $\nabla \cdot \hat{\tau} = 0$ from Eq. (2.4).

- Incompressibility (ρ = constant): Continuity equation (2.2) reduces to $\nabla \cdot \vec{u} = 0$.
- No body forces ($\vec{f} = 0$): No gravity or external forces.
- Steadiness ($\partial_t \vec{u} = 0$): Steady flow.
- No vorticity ($\nabla \times \vec{u} = 0$): Irrotational flow (except at singularities).

3.1.2 Simplified equations

In this case the simplified equations are:

1. Continuity (incompressibility) (2.2) reduces to:

$$\nabla \cdot \vec{u} = 0. \tag{3.1}$$

2. NS equation (2.4) reduces to Euler's equation (inviscid momentum):

$$\rho(\vec{u} \cdot \nabla)\vec{u} = -\nabla p. \tag{3.2}$$

3. Irrotationality condition (introducing the velocity potential ϕ via the Helmholtz theorem):

$$\nabla \times \vec{u} = 0 \quad \Rightarrow \quad \vec{u} = \nabla \phi = (\partial_x \phi, \partial_y \phi) = (u, v). \tag{3.3}$$

4. Bernoulli's equation (for steady, inviscid, irrotational flow):

$$p + \frac{1}{2}\rho|\vec{u}|^2 = p_\infty + \frac{1}{2}\rho U^2 = \text{constant}. \tag{3.4}$$

   It is a consequence of the first law of thermodynamic. We show its derivation in Appendix A. Here, p_∞ is static flow pressure far from any bodies, and U is the x-component of the flow velocity also far from any emersed bodies.

3.1.3 Free-stream velocity

Here, the free-stream velocity $U > 0$ is defined as the **undisturbed flow velocity** far from any boundaries or objects (sometimes U_∞):

$$\vec{U} = \lim_{r\to\infty} \vec{u}(r,\theta) = (U,0)^T, \tag{3.5}$$

where (r,θ) are the polar coordinates on the $2d$ xy plane. It represents the constant fluid speed in the absence of disturbances. Its direction aligned with the positive x-axis: $\vec{U} = (U,0)$, thus its magnitude is positive $U > 0$.

For potential flow around a cylinder with circulation Γ, the velocity components in polar coordinates (r,θ) are (see Sec. 3.3):

$$u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta, \tag{3.6}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \tag{3.7}$$

As distance from the cylinder increases ($r\to\infty$) one has (far-field behavior):

$$u_r \to U\cos\theta, \quad u_\theta \to -U\sin\theta. \tag{3.8}$$

Converting to Cartesian components (using rotational transfer matrix) one finds:

$$u = u_r\cos\theta - u_\theta\sin\theta \underset{r\to\infty}{\to} U\cos^2\theta + U\sin^2\theta = U, \tag{3.9}$$

$$v = u_r\sin\theta + u_\theta\cos\theta \underset{r\to\infty}{\to} U\cos\theta\sin\theta - U\sin\theta\cos\theta = 0, \tag{3.10}$$

thus confirming $\vec{u} \to (U,0)$. The physical significance of U is related to:

- The dynamic pressure via the Bernoulli's equation:

$$p_\infty + \frac{1}{2}\rho U^2 = \text{constant}, \tag{3.11}$$

- The lift force[1] (Kutta-Joukowski Theorem):

$$L = \rho U \Gamma, \tag{3.12}$$

as defined in Sec. 4.7, shows that the lift per unit length is directly proportional to U.

- Stagnation points, locations where $|\vec{u}| = 0$, depend on U:

$$\sin\theta_{\text{stag}} = \frac{\Gamma}{4\pi R U} \quad (\text{for } |\Gamma| \leq 4\pi R U). \tag{3.13}$$

- Reynolds[2] number[3]:

$$\text{Re} = \frac{\rho U D}{\mu}, \tag{3.14}$$

where ρ is the fluid density, D is the characteristic length scale (for a cylinder, this is its diameter), μ is the dynamic viscosity of the fluid.

3.1.4 Circulation

In fluid dynamics, circulation Γ is a measure of the "rotation" or "swirl" of a velocity field around a closed curve. Mathematically, it is defined by the Stokes integral:

$$\Gamma = \oint_C \vec{u} \cdot d\vec{l}, \tag{3.15}$$

where C is a closed contour (a loop) in the flow, $\vec{u}$ is the velocity vector field, $d\vec{l}$ is the line element tangent to the contour. It tells us how much the fluid circulates around C. By Stokes' theorem,

[1] Actually the lift force L per unit length is the force perpendicular to the free stream.

[2] Osborne Reynolds (1842–1912) was an Irish-born British engineer and physicist, best known for his pioneering experiments on the transition between laminar and turbulent flow in pipes. He introduced the dimensionless Reynolds number, which characterizes flow regimes and remains a cornerstone of fluid mechanics.

[3] Reynolds number compares inertial forces to viscous forces:

$$\text{Re} = \frac{\text{inertial effects}}{\text{viscous effects}}.$$

High Re: inertia dominates – flow tends to be turbulent. Low Re: viscosity dominates – flow tends to be laminar. For flow around a cylinder of diameter D in free stream velocity U: $\text{Re} < 1$ – creeping (Stokes) flow. For $10^3 \lesssim \text{Re} \lesssim 2\times10^5$ – laminar vortex shedding. For $\text{Re} \gtrsim 2 \times 10^5$ – transition to turbulence.

circulation is related to the vorticity $\vec{\omega} = \nabla \times \vec{u}$ inside the surface S spanned by C:

$$\Gamma = \iint_S (\nabla \times \vec{u}) \cdot \vec{n}\, dS = \iint_S \omega_n\, dS, \tag{3.16}$$

where ω_n is the component of vorticity normal to the surface. In potential (irrotational) flow, circulation is zero unless a vortex or a lifting object (like an airfoil) is present. For lift generation (Kutta–Joukowski theorem), circulation plays a central role (see Sec. 4.7):

$$L = \rho U \Gamma, \tag{3.17}$$

where L is the lift force per unit span length (force per meter), i.e. per unit length along the axis of the cylinder (or wing span).

3.2 Stream function vs. velocity potential

In order to describe the potential flow around a $2d$ circle (cylinder) one has to introduce the so-called **stream function** ψ and **velocity potential** ϕ. By definition one has:

- **Stream function** (ψ):
  - Defined for **incompressible flows** ($\nabla \cdot \vec{u} = 0$), i.e. ensures mass conservation.
  - Exists only in $2d$ (or axisymmetric $3d$ flows).
  - It is the imaginary part of the complex flow potential defined in Sec. 4.
- **Velocity potential** (ϕ):
  - Defined for **irrotational flows** ($\nabla \times \vec{u} = 0$, thus $\vec{u} = \nabla\phi$), i.e. ensures energy conservation.
  - Exists in $2d$ **and** $3d$.
  - Related to mechanical work of the flow: $\phi = \int \vec{u} \cdot d\vec{r}$, since $\vec{u} = \nabla\phi$.
  - It is the real part of the complex flow potential defined in Sec. 4.

The relation to velocity components is given by:

- $2d$ Cartesian coordinates (x, y):

Stream Function:	$u = \dfrac{\partial \psi}{\partial y},$	$v = -\dfrac{\partial \psi}{\partial x},$
Velocity Potential:	$u = \dfrac{\partial \phi}{\partial x},$	$v = \dfrac{\partial \phi}{\partial y}.$

- $2d$ polar coordinates (r, θ):

Stream Function:	$u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta},$	$u_\theta = -\dfrac{\partial \psi}{\partial r},$
Velocity Potential:	$u_r = \dfrac{\partial \phi}{\partial r},$	$u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}.$

The defining equations are:

	Stream Function (ψ)	**Velocity Potential** (ϕ)
Continuity ($\nabla \cdot \vec{u} = 0$)	Automatically satisfied: $\nabla \cdot \vec{u} \equiv 0$	Requires Laplace's equation: $\nabla^2 \phi = 0$
Irrotationality ($\nabla \times \vec{u} = 0$)	Requires Laplace's equation: $\nabla^2 \psi = 0$	Automatically satisfied: $\nabla \times \vec{u} \equiv 0$

The physical interpretation of these potentials is:

Stream Function (ψ)	**Velocity Potential** (ϕ)
Constant ψ: Defines streamlines (paths tangent to flow direction)	**Constant ϕ:** Defines equipotential lines (lines of constant mechanical work)
Boundary Condition: $\psi =$ constant on solid boundaries (no-penetration condition)	**Boundary Condition:** $\frac{\partial \phi}{\partial n} = 0$ on solid boundaries (no-penetration condition)

Orthogonality in potential flow. For incompressible and irrotational flows the streamlines (ψ = constant) and equipotential lines (ϕ = constant) form an orthogonal networks, i.e. $\nabla\phi\cdot\nabla\psi = 0$. This follows from the Cauchy-Riemann equations:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{3.18}$$

Example: Uniform flow U in x-direction (see Sec. 3.3):

$\psi = Uy,$	$\phi = Ux.$
Streamlines: y = constant	Equipotentials: x = constant

Example: Flow around cylinder with a radius R amd nonzero circulation Γ (see Sec. 3.3):

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta, \tag{3.19}$$

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln\frac{r}{R}. \tag{3.20}$$

3.3 Potential flow around a cylinder with circulation

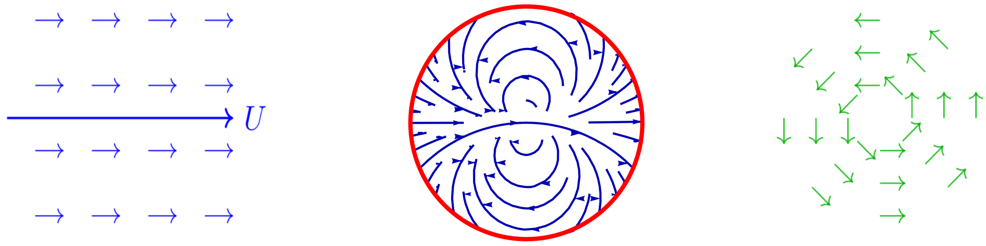
This is also known as the Kutta[1]-Joukowski[2] setup – a classical model of lift in inviscid, incompressible, irrotational flow around a cylinder or an airfoil, where circulation is introduced. It's the framework that leads to the Kutta-Joukowski theorem for lift (see Sec. 4.7).

[1]Martin Wilhelm Kutta (1867–1944) was a German mathematician and aerodynamicist. He introduced the Kutta condition (see Sec. 4.5), which determines how flow departs from the trailing edge of an airfoil, and co-formulated the Kutta–Joukowski lift theorem (see Sec. 4.7). He is also well known in mathematics for developing the Runge–Kutta methods for solving differential equations, still widely used in numerical analysis today.

[2]Nikolai Yegorovich Zhukovsky (1847–1921), often transliterated as Joukowski, was a Russian scientist regarded as the founder of Russian aerodynamics. He independently derived the lift theorem and pioneered the use of conformal mapping to transform circles into airfoil shapes, laying the foundation for modern airfoil theory. His institute in Moscow later became central to the development of Soviet aeronautics.

3.3.1 Flow decomposition

The flow is constructed by superimposing three types of simple flows: uniform flow + doublet flow[1] + vortex flow as shown in Figure 3.1.



(a) **(b)** **(c)**

Figure 3.1 **(a)** Uniform Flow $\vec{U} = (U, 0)$. **(b)** Doublet Flow (Cylinder). **(c)** Vortex Flow (Γ).

The total flow combines:

$$\vec{u}_{\text{total}} = \vec{u}_{\text{uniform}} + \vec{u}_{\text{doublet}} + \vec{u}_{\text{vortex}}, \tag{3.21}$$

where U scales the uniform and doublet components:

- Uniform flow (velocity u in x-direction):

$$\phi_{\text{uniform}} = Ux = Ur\cos\theta, \quad \psi_{\text{uniform}} = Uy = Ur\sin\theta. \tag{3.22}$$

- Doublet flow (represents the cylinder):

$$\phi_{\text{doublet}} = \frac{\kappa\cos\theta}{r}, \quad \psi_{\text{doublet}} = -\frac{\kappa\sin\theta}{r}, \tag{3.23}$$

  where $\kappa = UR^2$ ensures no flow penetration at $r = R$.

- Point vortex (adds circulation Γ):

$$\phi_{\text{vortex}} = \frac{\Gamma\theta}{2\pi}, \quad \psi_{\text{vortex}} = -\frac{\Gamma}{2\pi}\ln\frac{r}{R}. \tag{3.24}$$

[1] The doublet (or dipole) flow is inside the cylinder and therefore does not affect the external flow, but it ensures the no-penetration boundary condition on the cylinder surface.

- Combined potential and stream function:

$$\phi = \phi_{\text{uniform}} + \phi_{\text{doublet}} + \phi_{\text{vortex}} = U\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma\theta}{2\pi}, \tag{3.25}$$

$$\psi = \psi_{\text{uniform}} + \psi_{\text{doublet}} + \psi_{\text{vortex}} = U\left(r - \frac{R^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln\frac{r}{R}. \tag{3.26}$$

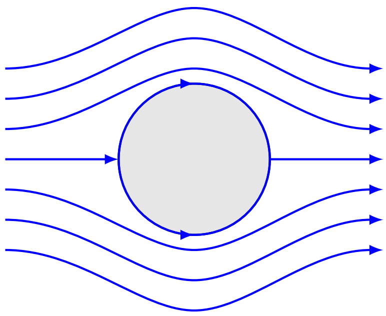
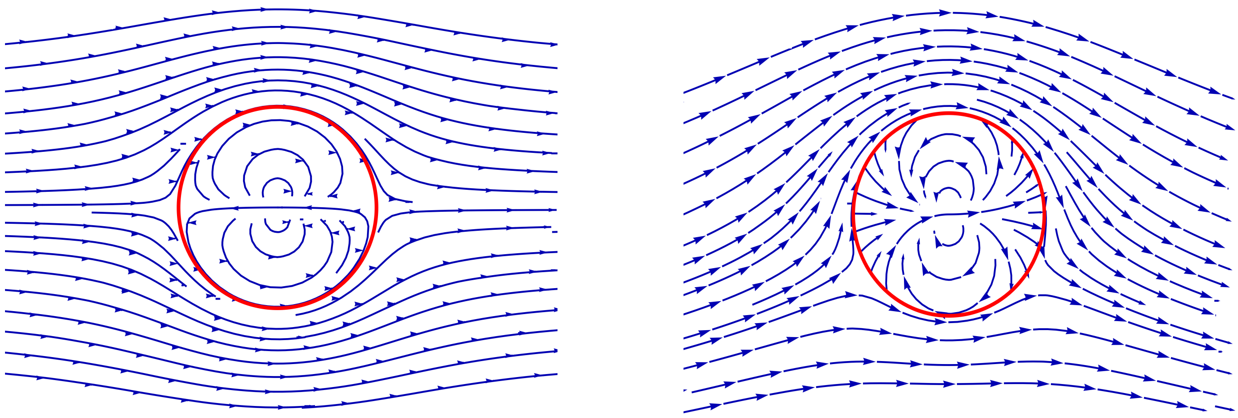


Figure 3.2 Streamlines of a steady flow around a circle.



(a) Uniform + doublet flow. **(b)** Uniform + doublet + circulation flow.

Figure 3.3 Flows without (a) and with circulation (b) (generated by Mathematica's StreamlinePlot[] command). The doublet (or dipole) flow is inside the cylinder and therefore does not affect the external flow, but it ensures the no-penetration boundary condition on the cylinder surface.

3.3.2 Velocity field

We can now compute the components of the velocity field in polar coordinates:

$$u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta, \tag{3.27}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \tag{3.28}$$

In Sec. 3.5 we show that they satisfy the simplified equations from Sec. 3.1.2 for steady, incompressible, inviscid, irrotational flow with circulation. On the cylinder surface ($r = R$) one has:

$$u_r = 0 \quad \text{(no flow through boundary)}, \tag{3.29}$$

$$u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi R}. \tag{3.30}$$

3.3.3 Pressure distribution

The pressure distribution comes from the Bernoulli's equation (3.4):

$$p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho|\vec{u}|^2 = \text{constant along a streamline}, \tag{3.31}$$

which can be solved with respect to the pressure of the fluid flow:

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho|\vec{u}|^2. \tag{3.32}$$

Here the magnitude of the flow velocity $|\vec{u}|$ is given by

$$|\vec{u}|^2 = u_r^2 + u_\theta^2 = U^2\cos^2\theta\left(\frac{R^2}{r^2} - 1\right)^2 + \left[\sin\theta\left(\frac{R^2 U}{r^2} + U\right) - \frac{\Gamma}{2\pi r}\right]^2. \tag{3.33}$$

where u_r and u_θ are defined in (3.27) and (3.28). On the cylinder ($r = R$) one has:

$$|\vec{u}|^2 = u_\theta^2 = \left(-2U\sin\theta + \frac{\Gamma}{2\pi R}\right)^2, \tag{3.34}$$

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho|\vec{u}|^2 = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta + \frac{\Gamma}{2\pi R}\right)^2. \tag{3.35}$$

Here $p = p(R, \theta)$ is the local pressure at the cylinder surface, and p_∞ is the static pressure far from the cylinder, where the velocity tends to the uniform free–stream U (the free-stream (reference) pressure).

3.3.4 Lift Force (Kutta-Joukowski Theorem)

Kutta-Joukowski lift. The lift force (per unit length) is the force perpendicular to the free stream (along x). For a circular cylinder $C(0, R)$ of radius R in a $2d$ flow, the force in the vertical direction y (lift) comes from integrating the pressure around the surface (see Sec. 4.7):

$$L = -\oint_C p \sin\theta \, R \, d\theta = -\int_0^{2\pi} p \sin\theta \, R \, d\theta \quad (y\text{-direction}). \tag{3.36}$$

Substituting p from (3.35) and integrating over θ one finds:

$$L = -\int_0^{2\pi} \left[p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho \left(-2U \sin\theta + \frac{\Gamma}{2\pi R} \right)^2 \right] \sin\theta \, R \, d\theta. \tag{3.37}$$

After simplification we arrive at the **Kutta-Joukowski formula** for the lift force:

$$L = \rho U \Gamma. \tag{3.38}$$

Magnus force. In addition, if the cylinder rotates uniformly with angular velocity ω, the circulation is give by:

$$\Gamma = \oint_C \vec{u} \cdot d\vec{l} = \int_0^{2\pi} u_\theta(R, \theta) \, R \, d\theta = 2\pi R^2 \omega, \tag{3.39}$$

where for a circular contour of radius R around the cylinder, the differential line element is tangential, $d\vec{l} = \hat{\mathbf{e}}_\theta R \, d\theta$, and the Cartesian components of the velocity are:

$$u = u_r \cos\theta - u_\theta \sin\theta, \quad v = u_r \sin\theta + u_\theta \cos\theta. \tag{3.40}$$

Here $u_\theta = \omega R$ and we also used the orthogonality of the polar basis vectors: $u_r \, \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$. This leads to the so-called Magnus lift force:

$$L = \rho U \Gamma = 2\pi \rho U R^2 \omega. \tag{3.41}$$

Drag force (D'Alembert's paradox). For inviscid potential flow, the drag force is zero:

$$D = -\oint p \cos\theta \, R \, d\theta = 0 \quad (x\text{-direction}). \tag{3.42}$$

This is D'Alembert's paradox, which states that there is no drag in ideal flow (viscosity is needed for drag).

3.4 Summary

Here we present all results for the flow around the cylinder in $2d$:

1. Velocity potential:
$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma\theta}{2\pi}. \tag{3.43}$$

2. Stream function:
$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln\left(\frac{r}{R}\right). \tag{3.44}$$

3. Surface velocity:
$$u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi R}. \tag{3.45}$$

4. Lift force (Kutta-Joukowski):
$$L = \rho U \Gamma. \tag{3.46}$$

5. Magnus force:
$$L = 2\pi\rho U R^2 \omega. \tag{3.47}$$

6. Drag force (D'Alembert's paradox):
$$D = 0. \tag{3.48}$$

This completes the traditional analytical analysis of potential flow around a cylinder with circulation, including lift, drag and Magnus force calculations.

3.5 Equations check

We show that the velocity components in polar coordinates from Eqs. (3.27) and (3.28):

$$u_r = \frac{\partial \phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta, \tag{3.49}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \tag{3.50}$$

satisfy the simplified equations from Section 3.1.2 for steady, incompressible, inviscid, irrotational flow with circulation.

1. Continuity (incompressibility). In polar coordinates the condition reads:

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0, \tag{3.51}$$

which is satisfied identically since (u_r, u_θ) are derived from a potential that solves Laplace's equation. A simple differentiation of u_r and u_θ also confirms this conclusion.

2. Euler's equation (inviscid momentum). It is given by:

$$\rho(\vec{u} \cdot \nabla)\vec{u} = -\nabla p. \tag{3.52}$$

For potential flow, this condition is automatically satisfied once p is defined through Bernoulli's equation as in Eq. (3.32):

$$p = p_\infty + \frac{\rho U^2}{2} - \frac{1}{2}\rho\left[U^2\cos^2\theta\left(\frac{R^2}{r^2} - 1\right)^2 + \left(\sin\theta\left(\frac{R^2 U}{r^2} + U\right) - \frac{\Gamma}{2\pi r}\right)^2\right]. \tag{3.53}$$

In this case:

$$\rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla p \quad \Longleftrightarrow \quad \begin{cases} \rho\left(u_r\dfrac{\partial u_r}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta^2}{r}\right) = -\dfrac{\partial p}{\partial r}, \\ \rho\left(u_r\dfrac{\partial u_\theta}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r u_\theta}{r}\right) = -\dfrac{1}{r}\dfrac{\partial p}{\partial \theta}, \end{cases}$$

where the pressure gradient is

$$\nabla p = \frac{\partial p}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\,\hat{\mathbf{e}}_\theta. \tag{3.54}$$

Inserting p, u_r and u_θ, one notes these equations are satisfied.

3. Irrotationality.

$$\nabla \times \vec{u} = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\big(r\,u_\theta\big) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 0. \tag{3.55}$$

Without circulation ($\Gamma = 0$), the flow is irrotational everywhere. For $\Gamma \neq 0$, the flow remains irrotational except at the origin, where the vortex singularity is located.

4. Bernoulli's equation. It is determined by:

$$p + \tfrac{1}{2}\rho|\vec{u}|^2 = p_\infty + \tfrac{1}{2}\rho U^2 = \text{constant along a streamline.} \tag{3.56}$$

Therefore, pressure is:

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho|\vec{u}|^2, \tag{3.57}$$

where $|\vec{u}|^2 = u_r^2 + u_\theta^2$ is shown explicitly in Eq. (3.33). Thus, the pressure distribution follows from the velocity field, with the constant fixed by the far-field condition $\vec{u} \to U\,\hat{\mathbf{e}}_x$ and $p \to p_\infty$ as $r \to \infty$.

Conclusion. The velocity components (u_r, u_θ) therefore constitute the classical potential flow solution around a cylinder with circulation, satisfying the simplified governing equations of incompressible, inviscid, irrotational flow.

Chapter 4

Joukowski Airfoil Theory

Chapter Objectives: We find the aerodynamic flow around airfoils using the Joukowski transformation, a complex-variable conformal mapping technique that transforms a circle into a symmetric or cambered airfoil shape. By analyzing potential flow around a circle and applying the Joukowski map, the behavior of ideal, inviscid flow around an airfoil can be studied analytically. We use the following references: [9–11].

4.1 Complex potential for ideal fluid flow in $2d$

We saw that for ideal potential (irrotational) flows the velocity potential ϕ and the streamfunction ψ fully defined the flow field basically replacing the velocity vector as the variable of interest. In arriving at a representation of the flow using complex variables we recall the standard representation of a complex variable:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}, \tag{4.1}$$

where a point in the $2d$ fluid plane is a complex number $z = x + iy$. Since both ϕ and ψ satisfy the Laplace equation $\nabla^2 f = 0$, they are harmonic functions. As such we saw that they they satisfy the Cauchy-Riemann conditions by:

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}, \tag{4.2}$$

where $\vec{u} = (u, v)$ are the Cartesian components of the velocity. Because of this ϕ and ψ are orthogonal to each other, $\nabla\phi \cdot \nabla\psi = 0$, and it is possible to use one or the other to represent the flow.

A new complex function (potential) can be defined, whose real and imaginary parts are the velocity potential and the streamfunction as:

$$\Phi(z) = \phi(z) + i\psi(z). \tag{4.3}$$

Thus, the derivative of Φ in terms of z is defined as:

$$W = \frac{\partial \Phi}{\partial z} = u - iv, \tag{4.4}$$

where $\vec{u} = (u, v)^T = (\partial_x\phi, \partial_y\phi)^T = (\partial_y\psi, -\partial_x\psi)^T$. The function $W(z)$ is called **complex velocity** of the flow. The explicit derivation of this formula is presented in Appendix B. The advantage of the complex variable formulations is that the function $W(z)$ allows us to use the full power of the complex analysis to describe flows in $2d$ [1, 12, 13].

It is often convenient to use cylindrical coordinates (r, θ) in two dimensions. The transformation from (x, y) to (r, θ) is given by:

$$u = u_r \cos\theta - u_\theta \sin\theta, \tag{4.5}$$

$$v = u_r \sin\theta + u_\theta \cos\theta. \tag{4.6}$$

The expression for the complex velocity W is then (by direct substitution and using Euler's identity, $\cos\theta \pm i \sin\theta = e^{\pm i\theta}$):

$$W = (u_r - iu_\theta)e^{-i\theta}. \tag{4.7}$$

4.2 Flow around a circle

The potential for a uniform flow with speed U around a circle of radius R centered at z_0 (its offset), with circulation Γ, is:

$$\Phi(z) = U\left((z - z_0) + \frac{R^2}{z - z_0}\right) - \frac{i\Gamma}{2\pi}\ln(z - z_0), \tag{4.8}$$

where z_0 is the center of the circle. In order to prove this formula, we start from polar coordinates representation of the velocity potential ϕ (3.43) and the stream function ψ (3.44) of the complex potential:

$$\Phi = \phi + i\psi = U\left[\left(r + \frac{R^2}{r}\right)\cos\theta + i\left(r - \frac{R^2}{r}\right)\sin\theta\right] + \frac{\Gamma\theta}{2\pi} - i\frac{\Gamma}{2\pi}\ln\frac{r}{R}. \tag{4.9}$$

Recall that $re^{i\theta} = r(\cos\theta + i\sin\theta) = z$ and $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$. Now look at the combination:

$$\left(r + \frac{R^2}{r}\right)\cos\theta + i\left(r - \frac{R^2}{r}\right)\sin\theta = \underbrace{r\cos\theta + ir\sin\theta}_{re^{i\theta}} + \frac{R^2}{r}\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}} = z + \frac{R^2}{z}. \tag{4.10}$$

That's the uniform flow + doublet flow in complex form. For the circulation terms one has:

$$\begin{aligned}\frac{\Gamma\theta}{2\pi} - i\frac{\Gamma}{2\pi}\ln\frac{r}{R} &= \frac{\Gamma}{2\pi}\left(\theta - i\ln\frac{r}{R}\right) = \frac{\Gamma}{2\pi}\Big(\theta - i(\ln r - \ln R)\Big) = \frac{\Gamma}{2\pi}(\theta - i\ln r + i\ln R) \\ &= \frac{1}{i}\frac{\Gamma}{2\pi}(\underbrace{i\theta + \ln r}_{\ln z} - \ln R) = -i\frac{\Gamma}{2\pi}\ln\frac{z}{R}.\end{aligned} \tag{4.11}$$

Note that since $z = re^{i\theta}$, then $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$. Now insert (4.10) an (4.11) back into (4.9) to find the complex flow potential around a circle centered in the origin ($z_0 = 0$):

$$\Phi(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln\frac{z}{R}. \tag{4.12}$$

To place the center of the circle at an arbitrary point z_0 we just shift the center $z \to z - z_0$:

$$\Phi(z) = U\left[(z - z_0) + \frac{R^2}{z - z_0}\right] - \frac{i\Gamma}{2\pi}\ln\frac{z - z_0}{R}. \tag{4.13}$$

Note (vortex rotation): For a vortex of strength Γ at the origin:

$$\phi_{\text{vortex}} = \pm\frac{\Gamma}{2\pi}\theta, \quad \psi_{\text{vortex}} = \mp\frac{\Gamma}{2\pi}\ln\frac{r}{R}, \tag{4.14}$$

where r, θ are polar coordinates. The sign of Γ indicates clockwise vs counterclockwise circulation. The $\pm$ in ϕ_{vortex} is tied to which direction you take θ as positive (usually counterclockwise). For the standard convention (counterclockwise positive circulation) the vortex potential is:

$$w_{\text{vortex}}(z) = \frac{i\Gamma}{2\pi} \ln z. \tag{4.15}$$

Then $\phi = -\frac{\Gamma}{2\pi}\theta$ and $\psi = \frac{\Gamma}{2\pi} \ln \frac{r}{R}$. Note that till now our working convention above was the clockwise circulation. We can write the total complex flow potential as:

$$\Phi(z) = U\left[(z - z_0) + \frac{R^2}{z - z_0}\right] \pm \frac{i\Gamma}{2\pi} \ln \frac{z - z_0}{R}. \tag{4.16}$$

4.3 Milne-Thomson circle theorem

Another mathematical way of arriving at (4.8) is the **Milne-Thomson circle theorem**. It is a fundamental result in fluid dynamics used to find the new fluid flow pattern when a circular cylinder is placed into a known two-dimensional, irrotational flow. Let the complex potential of an existing fluid flow be described by the analytic function $f(z)$, where $z = x + iy$. Suppose all singularities of $f(z)$ are outside the circle $|z| = R$. If a solid circular cylinder of radius R, centered at the origin, is introduced into this flow, the new complex potential, $\Phi(z)$, for the flow outside the cylinder is given by:

$$\Phi(z) = f(z) + \overline{f\left(\frac{R^2}{\bar{z}}\right)}. \tag{4.17}$$

Here, overline means complex conjugate. We leave this theorem without prove, however we show how it works for a uniform flow. Consider a uniform flow of speed U in the positive x-direction. The complex potential for this flow alone is:

$$\Phi_{\text{uniform}}(z) = Uz \equiv f(z). \tag{4.18}$$

Now, we want to introduce an impermeable circular cylinder of radius R centered at the origin. The boundary condition on the cylinder surface ($|z| = R$) is that there is no flow normal to the surface, which means the cylinder must be a streamline ($\psi =$ constant).

According to the Milne-Thomson circle theorem, if $f(z)$ is the complex potential for a flow in an infinite domain, and there are no singularities inside the circle $|z| = R$, then the complex potential for the same flow with an impermeable circular cylinder of radius R centered at the origin is given by Eq. (4.17). For uniform flow $f(z) = Uz$, thus

$$\Phi(z) = Uz + \overline{U\left(\frac{R^2}{\bar{z}}\right)} = Uz + U\frac{R^2}{z} = U\Big(z + \frac{R^2}{z}\Big), \tag{4.19}$$

where we use that U is a real constant, and $\overline{fg} = \bar{f}\bar{g}$. This is exactly the uniform flow given by Eq. (4.12), in the absence of circulation, for a circle $C(0, R)$ centered at the origin. Recall that the Uz term represents the uniform free stream flow, and the $U\frac{R^2}{z}$ term represents a doublet (or dipole) at the origin. This singularity is inside the cylinder and therefore does not affect the external flow, but it ensures the no-penetration boundary condition on the cylinder surface.

4.4 The Joukowski transformation

The transformation maps the z-plane (circle) to the ζ-plane (airfoil):

$$\zeta(z) = z + \frac{c^2}{z}. \tag{4.20}$$

To create an airfoil with a sharp trailing edge, the circle in the z-plane must pass through the point $z = c \in \mathbb{R}^+$. For a circle with center $z_0 = m + in$, where $m, n \in \mathbb{R}$, the radius R is determined by:

$$R = |c - z_0| = \sqrt{(c - m)^2 + n^2}. \tag{4.21}$$

The inverse transformation is[1]:

$$z(\zeta) = \frac{\zeta + \sqrt{\zeta^2 - 4c^2}}{2}. \tag{4.22}$$

The Joukowski transformation uses the parameters of the initial circle to determine the final shape of the airfoil. Here is how each parameter affects the geometry:

[1] The transformation maps the z-plane (circle) to the ζ-plane (airfoil). The circle is the input shape that is being acted upon by the function$\zeta(z)$. This circle is specifically designed to pass through the point $z = c$ in order to produce an airfoil with a sharp trailing edge. The center and radius of the circle are determined by the complex numbers $z_0 = m + in$ and $R = |c - z_0|$, respectively. Essentially, the circle is the original geometry that you are deforming. The transformation takes a simple shape (a circle) and, by using a complex function, maps its points to create a more complex, aerodynamically useful shape (an airfoil).

- The pole c: This value is a crucial point on the real axis (i.e. is a positive real number). It defines the location of the two singular points, or **poles**, of the transformation at $z = c$. The existence of these poles is what allows the transformation to create the sharp trailing edge of an airfoil. A key property of the Joukowski transformation is that the circle in the z-plane must pass through the point $z = c$. The pole at $z = c$ in the transformation maps the point on the circle to a cusp, or a sharp point, on the resulting airfoil. This cusp corresponds to the trailing edge of the airfoil. The magnitude of c scales the overall size of the airfoil.

- The number m (real part of the circle's center z_0) determines the airfoil's **camber**, or the curvature of the mean line. Increasing the value of m relative to c makes the airfoil more curved, which can increase lift at a given angle of attack.

- The number n (imaginary part of the circle's center z_0) controls the airfoil's **thickness**. Increasing the absolute value of n makes the airfoil thicker. A thicker airfoil can provide more structural strength and can be designed for higher lift.

- The parameter R (the radius of the circle) is not an independent variable but is derived from the other parameters using the equation $R = |c - z_0|$. It defines the size of the circle that is being transformed.

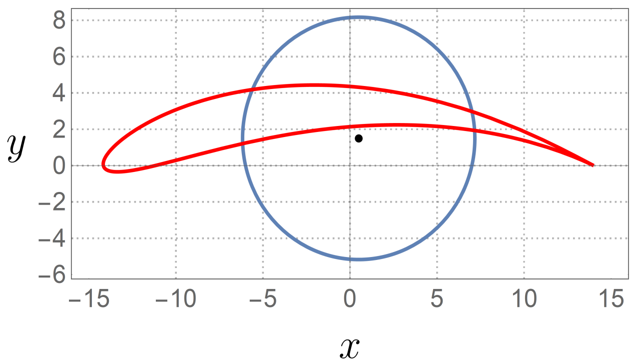


Figure 4.1 Joukowski transformation of a circle with radius $R = 6.67$ and parameters: $z_0 = (0.5, 1.5) = (m, n)$, $c = 7$ (see App. C for Mathematica code).

4.5 Kutta Condition

The Kutta condition states that the flow must leave the sharp trailing edge smoothly, which implies the velocity there is finite. We achieve this by setting the velocity to zero at the point $z = c$ in the circle plane. The complex velocity for the flow around the offset circle is:

$$W = \frac{d\Phi}{dz} = U\left(1 - \frac{R^2}{(z - z_0)^2}\right) - \frac{i\Gamma}{2\pi(z - z_0)}. \tag{4.23}$$

Set $\frac{dW}{dz} = 0$ at $z = c$:

$$U\left(1 - \frac{R^2}{(c - z_0)^2}\right) - \frac{i\Gamma}{2\pi(c - z_0)} = 0. \tag{4.24}$$

Using $R^2 = |c - z_0|^2 = (c - z_0)(c - z_0)^*$, where $*$ denotes the complex conjugate, one has:

$$U\left(1 - \frac{(c - z_0)(c - z_0)^*}{(c - z_0)^2}\right) = \frac{i\Gamma}{2\pi(c - z_0)} \quad \Rightarrow$$

$$U\left(1 - \frac{(c - z_0)^*}{(c - z_0)}\right) = \frac{i\Gamma}{2\pi(c - z_0)} \quad \Rightarrow$$

$$U\big((c - z_0) - (c - z_0)^*\big) = \frac{i\Gamma}{2\pi} \tag{4.25}$$

We can deal with the expression in the brackets by substituting $z_0 = m + in$:

$$(c - z_0) - (c - z_0)^* = ((c - m) - in) - ((c - m) + in) = -2in. \tag{4.26}$$

Inserting back to (4.25) and solving for the circulation Γ one finds:

$$\Gamma = -4\pi n U. \tag{4.27}$$

This shows that the circulation required for smooth flow is directly proportional to the camber parameter n. Note that the lift now is (the minus sign accounts for the clockwise circulation):

$$L = -\rho U \Gamma = 4\pi\rho n U^2. \tag{4.28}$$

In order to show that this is the result for the lift, we need the Blasius theorem.

4.6 The Blasius theorem

The **Blasius Theorem** is a fundamental result in potential flow theory that relates the hydrodynamic force (lift and drag) on a two-dimensional body to the complex velocity potential of the flow around it. It is particularly useful in aerodynamics for calculating forces on airfoils without viscosity. It states that for a steady, incompressible, irrotational (potential) $2d$ flow around a body, the net force (per unit span) acting on the body is given by the contour integral:

$$F_x - iF_y = \frac{i\rho}{2} \oint \left(\frac{d\Phi}{dz} \right)^2 dz, \tag{4.29}$$

where F_x is the force component in the x-direction (drag, in the absence of viscosity, should be zero), F_y is the force component in the y-direction (lift), ρ is the fluid density, and $\Phi(z)$ is the complex flow potential. The integral is taken **counterclockwise** around the body (in case of a **clockwise** circulation there is an additional minus sign).

The theorem is derived from Bernoulli's equation and the Cauchy residue theorem.

1. **Pressure Force on the Body**: The force acting on the body is due to pressure p. The infinitesimal force components on a small segment dz of the contour are:

$$dF_x = -p\,dy, \quad dF_y = p\,dx. \tag{4.30}$$

Thus, the total force (per unit span) is:

$$F_x = -\oint p\,dy, \quad F_y = \oint p\,dx. \tag{4.31}$$

In complex form one has:

$$F_x - iF_y = -\oint p(dy + idx) = -i \oint p\,dz^*, \tag{4.32}$$

where $dz^* = dx - idy$ is the complex conjugate of dz.

2. **Bernoulli's Equation**: For steady, inviscid, incompressible flow, Bernoulli's equation gives:

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho|\vec{u}|^2, \tag{4.33}$$

   where $|\vec{u}|^2 = u^2 + v^2 = \left|\frac{d\Phi}{dz}\right|^2$. Since p_∞ and $\frac{1}{2}\rho U^2$ are constant, they integrate to zero around a closed contour, leaving:

$$F_x - iF_y = \frac{i\rho}{2}\oint \left|\frac{d\Phi}{dz}\right|^2 dz. \tag{4.34}$$

3. **Complex Velocity Squared**: Using $\left|\frac{d\Phi}{dz}\right|^2 = \left(\frac{d\Phi}{dz}\right)\left(\frac{d\Phi}{dz}\right)^*$, but since $\frac{d\Phi}{dz}$ is analytic outside the body, we can evaluate it using residues. For large $|z|$, the dominant term is often a vortex or doublet contribution:

$$\frac{d\Phi}{dz} \approx U + \frac{i\Gamma}{2\pi z} + \ldots \tag{4.35}$$

   Squaring and applying the residue theorem leads to the Kutta-Joukowski lift formula, as we show below.

4.7 Lift and Drag (Kutta-Joukowski Theorem)

The aerodynamic force is calculated with the Blasius formula. For simplicity, we perform the integral around a circle at the origin, as the net force is independent of the offset:

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{d\Phi}{dz}\right)^2 dz. \tag{4.36}$$

The potential for a circle at the origin is (counterclockwise circulation):

$$\Phi(z) = U\Big(z + \frac{R^2}{z}\Big) + \frac{i\Gamma}{2\pi}\ln\frac{z}{R}. \tag{4.37}$$

The derivative is:

$$\frac{d\Phi}{dz} = U\left(1 - \frac{R^2}{z^2}\right) + \frac{i\Gamma}{2\pi z}. \tag{4.38}$$

Squaring this expression one finds:

$$\left(\frac{d\Phi}{dz}\right)^2 = \left[U\left(1-\frac{R^2}{z^2}\right)+\frac{i\Gamma}{2\pi z}\right]^2 = U^2\left(1-\frac{2R^2}{z^2}+\frac{R^4}{z^4}\right)+\frac{2Ui\Gamma}{2\pi z}\left(1-\frac{R^2}{z^2}\right)-\frac{\Gamma^2}{4\pi^2 z^2}. \tag{4.39}$$

By the Residue Theorem, the integral $\oint_C f(z)dz$ is $2\pi i \sum \text{Res}(f, z_k)$. The only pole inside the contour C is at $z = 0$. We need the coefficient of the z^{-1} term of $(\frac{d\Phi}{dz})^2$. The only term contributing to this is the cross-term:

$$\frac{2Ui\Gamma}{2\pi z}\left(1-\frac{R^2}{z^2}\right) = \frac{iU\Gamma}{\pi z} - \frac{iU\Gamma R^2}{\pi z^3}. \tag{4.40}$$

The residue is the coefficient of the $1/z$ term:

$$\text{Res}\left[\left(\frac{d\Phi}{dz}\right)^2, z=0\right] = \frac{iU\Gamma}{\pi}. \tag{4.41}$$

Now, we evaluate the integral:

$$\oint_C \left(\frac{dW}{dz}\right)^2 dz = 2\pi i\left(\frac{iU\Gamma}{\pi}\right) = -2U\Gamma. \tag{4.42}$$

Substitute this result back into the Blasius formula (4.29) one finds:

$$F_x - iF_y = \frac{i\rho}{2}(-2U\Gamma) = -i\rho U\Gamma. \tag{4.43}$$

Equating the real and imaginary parts one finally gets (for counterclockwise circulation):

$$F_x = 0 \quad \text{(Drag)}, \tag{4.44}$$

$$F_y \equiv L = \rho U\Gamma \quad \text{(Lift)}. \tag{4.45}$$

For clockwise circulation the lift is $L = -\rho U\Gamma$. This is a bit confusing sometimes. The lift L is the force exerted on the body by the fluid, which is the reaction force to F_y. Thus, $L = -F_y$ by some conventions, or we can define F_y as lift itself. If we define lift L as the positive upward force on the airfoil, and the circulation Γ is positive (counter-clockwise), then the force is upward.

Chapter 5

Effective Description of the Airfoil

Chapter Objectives: We discuss the main characteristics of the airfoil and present some simulated models of a wing on Boeing 737-800 airplane.

5.1 Basics of airfoils

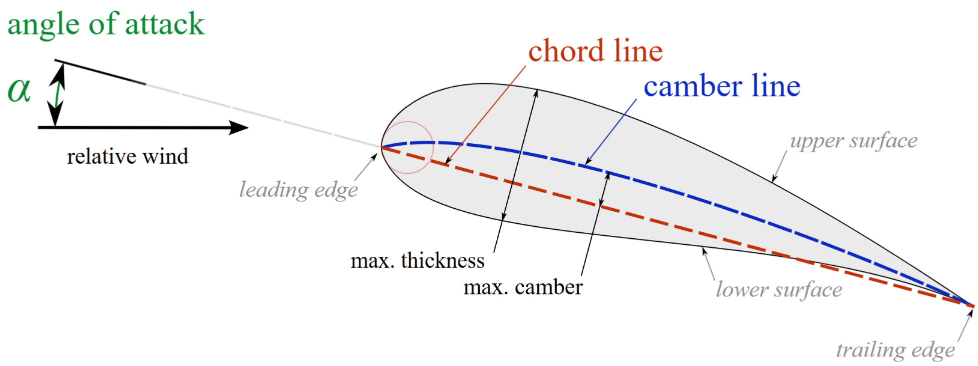


Figure 5.1 Basic characteristics of the airfoil (Wikipedia).

An airfoil is a specially designed cross-sectional shape of an aircraft wing, tail, or blade that enables the generation and control of aerodynamic forces during flight. Its geometry is carefully engineered to create a pressure difference between the upper and lower surfaces as air flows past, producing lift to support the aircraft's weight while also influencing drag, stability, and overall efficiency. The basic characteristics of the airfoil are [11, 14, 15]:

- **Leading Edge**: The leading edge is the forward-most part of an airfoil, the point that first meets the oncoming airflow. Its shape is critical because it determines how smoothly the air can begin flowing over the upper and lower surfaces. A well-rounded leading edge helps delay flow separation and reduces the risk of stalling, while sharper designs can improve efficiency at specific flight conditions. By controlling how the air initially divides, the leading edge plays a fundamental role in the generation of lift and in the overall aerodynamic performance of the airfoil.
- **Trailing Edge**: The trailing edge is the rearmost part of an airfoil, where the airflow that has traveled along the upper and lower surfaces comes together and leaves the wing. Its design strongly influences how smoothly the two streams of air merge, which in turn affects lift, drag, and overall aerodynamic efficiency. A sharp trailing edge helps guide the flow cleanly downstream, minimizing turbulence and drag, while also determining how circulation around the airfoil is established according to the Kutta condition. In many cases, movable control surfaces such as ailerons, elevators, or flaps are attached near the trailing edge, allowing pilots to adjust lift, maneuverability, and stability during flight.
- **Chord Line**: The chord line is an imaginary straight line that connects the leading edge to the trailing edge of an airfoil. It serves as a fundamental reference in aerodynamics, providing a baseline for measuring key geometric and aerodynamic parameters. Most importantly, it is used to define the angle of attack, which is the angle between the chord line and the oncoming airflow (relative wind). The chord line is also essential in describing the camber (the curvature of the airfoil) and the thickness distribution. Because of its role as a reference axis, the chord line is central to analyzing and predicting the lift, drag, and stability characteristics of an airfoil.
- **Camber**: Camber refers to the curvature of an airfoil's surfaces, typically measured as the deviation of the mean camber line (the curve midway between the upper and lower surfaces) from the chord line. It is one of the most important geometric features influencing how an airfoil generates lift. A positively cambered airfoil has a more pronounced curvature on the upper surface and a flatter lower surface, which accelerates airflow above the wing, reduces pressure, and produces greater lift. Conversely, a symmetrical airfoil (zero camber) generates little or no lift at zero angle of attack but can still produce lift when tilted relative to the oncoming airflow. The degree and distribution of

camber strongly affect not only lift but also drag, stall behavior, and aerodynamic efficiency, making it a central parameter in wing design.

- **Upper Surface**: The upper surface of an airfoil is typically curved and plays a crucial role in the production of lift. As air flows over this surface, the curvature causes the airflow to accelerate, which according to Bernoulli's principle results in a region of lower pressure compared to the air moving beneath the wing. This pressure difference between the upper and lower surfaces is a primary contributor to lift generation. The exact shape of the upper surface—its smoothness, degree of curvature (camber), and thickness distribution – greatly influences the magnitude of lift, drag characteristics, and stall behavior of the airfoil.
- **Lower Surface**: The lower surface of an airfoil is generally flatter than the upper surface, which helps maintain relatively higher pressure as air flows beneath the wing. This higher pressure, combined with the lower pressure generated over the curved upper surface, creates a pressure differential that produces lift. The thickness and shape of the lower surface also influence the overall structural strength of the wing, the aerodynamic drag, and the onset of flow separation at high angles of attack. In many airfoil designs, subtle shaping of the lower surface is used to optimize lift-to-drag ratios and ensure stable airflow under varying flight conditions.
- **Angle of Attack**: The angle of attack (AoA) is defined as the angle between the chord line of an airfoil and the direction of the oncoming airflow, often referred to as the relative wind. By adjusting this angle, pilots can directly control the lift and drag forces acting on the wing. Increasing the angle of attack generally increases lift up to a certain point, but if it becomes too steep, the airflow can separate from the upper surface, leading to a stall and a sudden loss of lift. Proper management of the angle of attack is therefore critical during key flight operations such as takeoff, landing, and maneuvering, as it balances the need for lift with aerodynamic efficiency and safety.
- **Thickness**: The thickness of an airfoil refers to the distance between its upper and lower surfaces, usually measured perpendicular to the chord line. Thickness is a critical factor in both aerodynamic performance and structural design. Thicker airfoils can generate more lift due to increased surface area and favorable pressure distribution, but they often experience higher drag, which can reduce overall efficiency. Conversely, thinner airfoils produce less drag and may be better suited for high-speed flight but generate less lift and may stall more abruptly. The distribution of thickness along the chord, not just the maximum thickness, also affects airflow behavior, lift characteristics,

and stall performance, making it an essential consideration in airfoil design.

- **Camber Line**: The camber line is an imaginary line drawn midway between the upper and lower surfaces of an airfoil, effectively representing the average curvature of the wing. It serves as a key reference for quantifying the amount and distribution of camber in the airfoil's design. The shape of the camber line determines how the airfoil generates lift: a more pronounced curvature typically produces greater lift at lower angles of attack, while a flatter camber line results in more neutral lift characteristics, as seen in symmetric airfoils. By analyzing the camber line, aerodynamicists can predict pressure distribution, optimize lift-to-drag ratios, and assess stall behavior, making it a fundamental tool in wing design.

- **Lift Force**: Lift is the aerodynamic force that acts perpendicular to the oncoming airflow, allowing an aircraft to become airborne and remain in flight. It is primarily generated by the airfoil's shape, especially its curvature (camber) and angle of attack, which create a pressure difference between the lower and upper surfaces. Faster airflow over the upper surface produces lower pressure, while slower airflow beneath maintains higher pressure, resulting in an upward force. The magnitude of lift depends on factors such as airspeed, air density, wing area, and the airfoil's geometry. Proper management of lift is essential for takeoff, cruising, maneuvering, and landing, as it directly counteracts the aircraft's weight.

- **Drag Force**: Drag is the aerodynamic force that opposes the motion of an aircraft through the air, acting parallel and opposite to the relative airflow. It arises from air resistance against the aircraft's surfaces, including the airfoil, and is influenced by factors such as airspeed, air density, surface roughness, and the shape of the wing. Airfoils contribute to drag in two main ways: form drag, caused by the shape of the airfoil, and skin friction, caused by friction between the air and the surface. Pilots can indirectly control drag by adjusting the angle of attack – the angle between the chord line of the airfoil and the oncoming airflow – which simultaneously affects lift. Proper management of drag is essential during takeoff, landing, and maneuvers to maintain efficiency, stability, and safe operation.

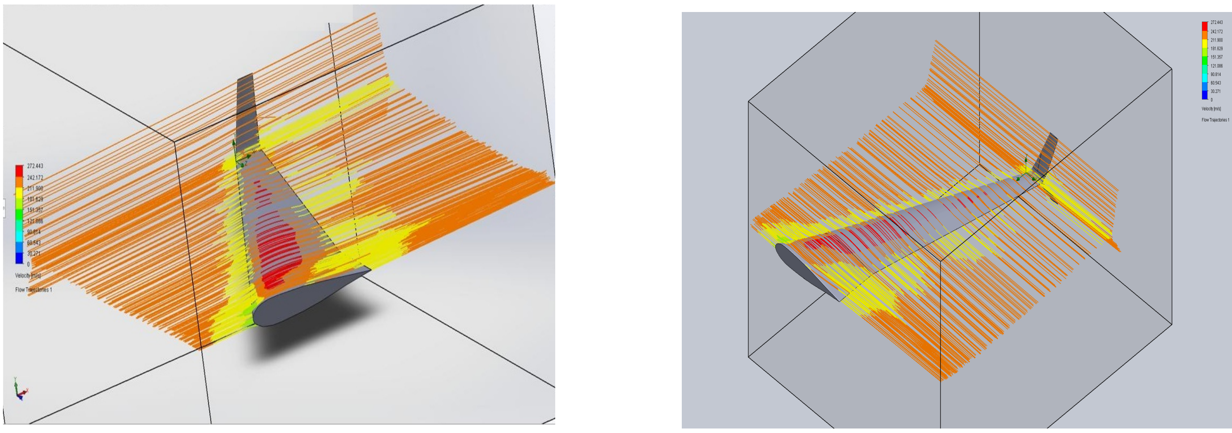
5.2 Models of the wing of Boeing 737-800

To demonstrate the applicability of fluid mechanics, we present velocity (Fig. 5.2) and pressure (Fig. 5.3) distribution models around the wing of a Boeing 737-800. The flow fields were generated using SOLIDWORKS simulations. The initial conditions correspond to a flight altitude of 9000 m, with a freestream velocity of approximately $U = 227\,\mathrm{m/s}$. At this altitude, the ambient pressure is about one-third of the standard atmospheric pressure, and the temperature is 0°C. Both models are at the same angle of attack.

The simulations illustrate that regions of higher airflow velocity (yellow–red in Fig. 5.2) correspond to lower pressure zones (cyan–blue in Fig. 5.3). This inverse relationship between velocity and pressure is a direct consequence of Bernoulli's principle, which states that along a streamline, an increase in fluid speed is accompanied by a decrease in static pressure.

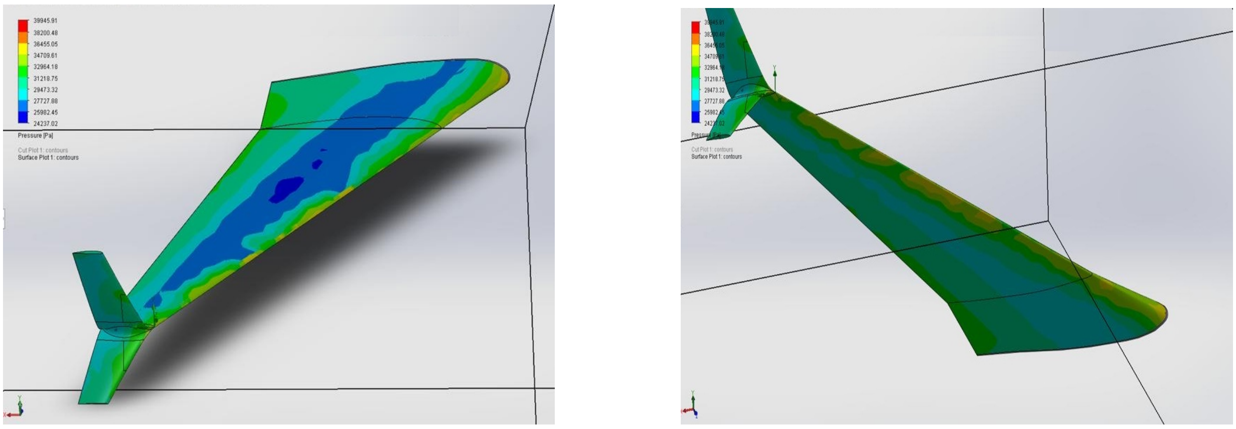
As a result, the pressure on the upper surface of the wing is lower than that on the lower surface, generating a net upward force known as lift. The greater the velocity difference between the upper and lower surfaces, the larger the pressure difference, and hence the greater the lift. This effect is also consistent with the Kutta-Joukowski theorem, which relates lift per unit span to the circulation of the flow around the airfoil.

In summary, the models demonstrate the fundamental aerodynamic mechanism of lift: the wing geometry shapes the flow so that high-velocity, low-pressure regions develop over the wing, producing the net upward force that enables sustained flight.



(a) Velocity around Boeing 737-800 (side). **(b)** Velocity around Boeing 737-800 (top).

Figure 5.2 **(a)** Velocity field flow around Boeing 737-800 wing from the side of the airplane. **(b)** Velocity field flow around Boeing 737-800 wing on the top of the airplane. The models are generated by SOLIDWORKS software. The free velocity of the flow at height of $9000\,m$ is $U = 227\,m/s$. The pressure at that height is 1/3 of the standard atmosphere. Temperature is 0° Celsius.



(a) Pressure for Boeing 737-800 (top). **(b)** Pressure for Boeing 737-800 (below).

Figure 5.3 **(a)** Pressure field flow around Boeing 737-800 wing from the top of the wing. **(b)** Pressure field flow around Boeing 737-800 wing from below of the wing. The models are generated by SOLIDWORKS software. The free velocity of the flow at height of $9000\,m$ is $U = 227\,m/s$. The pressure at that height is 1/3 of the standard atmosphere. Temperature is 0° Celsius.

Chapter 6

Conclusions

In this thesis, we investigated the foundations of fluid mechanics and their application to aerodynamic lift generation. Starting from the conservation laws of mass, energy, and momentum, we derived the governing equations of fluid flow under simplified assumptions. Within the Kutta-Joukowski framework, we analyzed two-dimensional flow past a circular cylinder, obtaining explicit results for the velocity field, pressure distribution, and aerodynamic forces. Extending this framework through the Joukowski transformation, we demonstrated how airfoil geometries can be generated from circular profiles and studied the corresponding potential flow solutions.

To connect theory with practice, we modeled the velocity and pressure distributions around a Boeing 737-800 wing. The simulations supported the theoretical analysis, showing roughly that regions of higher velocity correspond to lower pressure and thus generate lift. This agreement illustrates the enduring relevance of classical fluid mechanics in describing both idealized systems and practical engineering applications.

Looking ahead, future work can proceed along two complementary directions. On the theoretical side, incorporating viscous effects and boundary layer theory would provide a more realistic description of lift, drag, separation, and other aerodynamic phenomena. Extensions to three-dimensional, compressible, and turbulent flows would further enrich the analytical framework. On the engineering side, high-fidelity computational fluid dynamics (CFD) simulations and wind tunnel experiments could refine the study of real wing geometries, including modern design features such as winglets[1], high-lift devices, and blended wing bodies[2].

[1]Winglets are small vertical or angled extensions at the tips of an aircraft's wings that reduce drag and improving fuel efficiency and range.

[2]Blended wing body (BWB) is an aircraft design where the fuselage and wings form a single smoothly integrated

In conclusion, this work demonstrates how classical fluid mechanics, combined with mathematical tools such as complex analysis, provides both a rigorous theoretical foundation and a practical framework for understanding and improving aerodynamic performance, while also opening avenues for future research at the intersection of theory and engineering practice.

shape, improving lift-to-drag ratio, fuel efficiency, and internal volume compared to conventional tube-and-wing designs.

Appendix A

Derivation of Bernoulli equation

We will derive Bernoulli's equation rigorously from the fundamental equations of fluid mechanics, explaining each physical and mathematical step.

Start with the Navier-Stokes momentum equation for an incompressible, viscous flow:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{f}. \tag{A.1}$$

For inviscid flow ($\mu = 0$) and in the absence of external body forces ($\vec{f} = 0$), this reduces to the Euler equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p. \tag{A.2}$$

Let us now rewrite the acceleration (convective) term using the following vector identity:

$$(\vec{u} \cdot \nabla)\vec{u} = \nabla \left(\frac{|\vec{u}|^2}{2} \right) - \vec{u} \times (\nabla \times \vec{u}). \tag{A.3}$$

This splits the acceleration into: a kinetic energy gradient $\nabla(|\vec{u}|^2/2)$ and a vorticity-dependent term $\vec{u} \times (\nabla \times \vec{u})$. Now, assume an irrotational flow ($\nabla \times \vec{u} = 0$). Substituting into Euler's equation one finds:

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{|\vec{u}|^2}{2} \right) = -\frac{1}{\rho}\nabla p. \tag{A.4}$$

Recall the velocity potential ($\vec{u} = \nabla \phi$), thus:

$$\frac{\partial}{\partial t}(\nabla \phi) + \nabla \left(\frac{|\nabla \phi|^2}{2} \right) = -\frac{1}{\rho}\nabla p. \tag{A.5}$$

Since time and space derivatives commute ($\nabla\partial_t\phi = \partial_t\nabla\phi$), this becomes:

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + \frac{p}{\rho}\right) = 0. \tag{A.6}$$

A gradient of zero implies the argument is spatially constant:

$$\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + \frac{p}{\rho} = C(t), \tag{A.7}$$

where $C(t)$ is a time-dependent constant (uniform in space). For a steady flow ($\partial_t\phi = 0$), thus:

$$\boxed{\frac{p}{\rho} + \frac{1}{2}|\vec{u}|^2 = \text{constant.}} \tag{A.8}$$

One can also include gravity, if it is significant, by just adding the potential energy term $\rho g z$ to the pressure:

$$\frac{p}{\rho} + \frac{1}{2}|\vec{u}|^2 + gz = \text{constant.} \tag{A.9}$$

Appendix B

Derivation of the Complex Velocity

B.1 Wirtinger derivatives

Let's first derive the Wirtinger derivatives from the change of variables $x, y \leftrightarrow z, \bar{z}$:

$$z = x + iy, \quad \bar{z} = x - iy \quad \Rightarrow \quad x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}. \tag{B.1}$$

We can treat z and $\bar{z}$ as independent variables and apply the chain rule:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}. \tag{B.2}$$

Differentiate the expressions for x, y with respect to z (holding $\bar{z}$ fixed):

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \qquad \frac{\partial y}{\partial z} = \frac{1}{2i} = -\frac{i}{2}. \tag{B.3}$$

This leads to:

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big). \tag{B.4}$$

Similar calculations follow for the conjugate operator $\partial/\partial\bar{z}$:

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y}, \tag{B.5}$$

with

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2}, \tag{B.6}$$

which leads to:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{B.7}$$

So the final Wirtinger derivatives are:

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{B.8}$$

B.2 Derivation of the complex velocity potential

Start with:

$$\Phi(x, y) = \phi(x, y) + i\psi(x, y), \quad z = x + iy. \tag{B.9}$$

Apply the Wirtinger operator:

$$\frac{\partial \Phi}{\partial z} = \frac{1}{2}\Big(\frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y}\Big). \tag{B.10}$$

Compute $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$:

$$\frac{\partial \Phi}{\partial x} = \phi_x + i\psi_x, \quad \frac{\partial \Phi}{\partial y} = \phi_y + i\psi_y. \tag{B.11}$$

Now apply $\partial/\partial z$:

$$\frac{\partial \Phi}{\partial z} = \frac{1}{2}\Big[(\phi_x + i\psi_x) - i(\phi_y + i\psi_y)\Big] = \frac{1}{2}\Big[\phi_x + i\psi_x - i\phi_y + \psi_y\Big]. \tag{B.12}$$

Next, we use the Cauchy–Riemann relations for an analytic Φ:

$$\phi_x = \psi_y, \qquad \phi_y = -\psi_x. \tag{B.13}$$

Substitute them back into the bracket to obtain:

$$\frac{\partial \Phi}{\partial z} = \frac{1}{2}\Big[\phi_x + i(-\phi_y) - i\phi_y + \phi_x\Big] = \frac{1}{2}\Big[2\phi_x - 2i\phi_y\Big] = \phi_x - i\phi_y. \tag{B.14}$$

Finally, we replace $\phi_x = u$ and $\phi_y = v$:

$$\frac{\partial \Phi}{\partial z} = u - iv. \tag{B.15}$$

Appendix C

The Joukowski Transformation

This chapter provides a demonstration of the Joukowski transformation using *Mathematica*. The transformation maps a circle in the complex z-plane to an airfoil in the ζ-plane, where the shape of the airfoil is determined by the properties of the initial circle. The transformation is defined by the equation:

$$\zeta(z) = z + \frac{c^2}{z}. \tag{C.1}$$

A key requirement for generating an airfoil with a sharp trailing edge is that the circle must pass through the point $z = c$, where c is a positive real number.

C.1 Mathematica Code

The following code plots both the original circle and the resulting airfoil, illustrating the effect of the transformation on Figure 4.1.

```
(* Define the parameters for the transformation *);

c = 7; (* the length of the airfoil is approximately 2c *);

m = 0.5 (* the approximate thickness of the airfoil *);

n = -1.5 (* the approximate curved shape of the airfoil *);

z0 = m + I*n;

(* Calculate the radius of the circle, which must pass through z = c *)
R = Abs[c - z0];
```

```
(* Define the complex variable for the circle *)
z[theta_] = z0 + R*Exp[I*theta];

(* Define the Joukowski transformation *)
joukowski[z_] = z + c^2/z;

(* Apply the transformation to the circle *)
zeta[theta_] = joukowski[z[theta]];

(* Plot the original circle *)
circlePlot = ParametricPlot[{Re[z[theta]], Im[z[theta]]}, {theta, 0, 2*Pi},
  PlotStyle -> {Thick, Blue},
  PlotLabel -> "Joukowski Transformation",
  AspectRatio -> Automatic];

(* Plot the resulting airfoil *)
airfoilPlot = ParametricPlot[{Re[zeta[theta]], Im[zeta[theta]]}, {theta, 0, 2*Pi},
  PlotStyle -> {Thick, Red}];

(* Show both plots together *)
Show[circlePlot, airfoilPlot,
  AxesLabel -> {"x", "y"},
  PlotRange -> All,
  ImageSize -> Medium]
```

C.2 Explanation of the Code

- **Defining Parameters:** The code sets the values for the pole c and the center of the circle $z_0 = m + in$. The radius R is automatically calculated to ensure the circle passes through $z = c$.

- **Defining the Circle:** The complex variable $z(\theta) = z_0 + Re^{i\theta}$ is used to represent the points on the circle parametrically.

- **Defining the Transformation:** The function `joukowski[`z`]` is a direct implementation of the transformation equation.

- **Applying the Transformation:** The transformation is applied to the circle's parametric equation, resulting in the function $\zeta(\theta)$.

- **Plotting:** The *Mathematica* function ‘ParametricPlot‘ is used to plot the real and imaginary parts of both the original circle and the resulting airfoil. The ‘Show‘ command is then used to display both plots on the same graph for comparison.

References

(1) Bertin, J. J.; Cummings, R. M., *Aerodynamics for Engineers*, 5th; Pearson Prentice Hall: 2009.

(2) Houghton, E. L.; Carpenter, P. W.; Collicott, S. H.; Valentine, D., *Aerodynamics for Engineering Students*, 6th; Elsevier: 2012.

(3) Phillips, W. F., *Mechanics of Flight*; John Wiley & Sons: 2004.

(4) Anderson, J. D., *Fundamentals of Aerodynamics*; McGraw-Hill: 2007.

(5) Fox, R. W.; McDonald, A. T., *Introduction to Fluid Mechanics*, 10th; Wiley: 2020.

(6) White, F. M., *Fluid Mechanics*, 9th; McGraw-Hill Education: 2022.

(7) Chassaing, P., *Fundamentals of Fluid Mechanics: For Scientists and Engineers*; CRC: 2023.

(8) Durst, F., *Fluid Mechanics: An Introduction to the Theory of Fluid Flows*; Springer: 2022.

(9) Cantwell, B. Two Dimensional Airfoil Theory, Lecture notes, AA200 Applied Aerodynamics, Stanford University.

(10) Dunkel, J. Classical Airfoil Theory, Lecture notes, MIT, 2014.

(11) Leishman, J. G., *Introduction to Aerospace Flight Vehicles*; Eagle Publishing: 2022.

(12) Auld, D.; Srinivas, *2-D Thin Aerofoil Theory*; University of Sydney: 1995.

(13) Babinsky, H. *Physics Education* **2003**, *38*, 497–503.

(14) Petrilli, J. L.; Paul, R. C.; Gopalarathnam, A.; Frink, N. T. In *AIAA Aviation Forum, Session: Airfoil/Wing/Configuration Aerodynamics II*; Published Online: 22 June 2013, 2013.

(15) He, X.; Li, J.; Mader, C. A.; Yildirim, A.; Martins, J. R. R. A. *Aerospace Science and Technology* **2019**, 8748–8761.